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A Galerkin symmetric and direct BIE method for Kirchhoff elastic plates: formulation and implementation¹

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SUMMARY

A variational Boundary Element formulation is proposed for the solution of the elastic Kirchhoff plate bending problem. The stationarity conditions of an augmented potential energy functional are first discussed. After addressing the topic of the choice of the test functions a regularization process based on integrations by parts is developed, which allows to express the formulation in terms of double integrals, the inner being at most weakly singular and the outer regular. Standard integration procedures may then be applied for their numerical evaluation in presence of both straight and curved boundaries. The normal slope and the vertical displacement must be C^0 and C^1 continuous respectively. Numerical examples show, through comparisons with analytical solutions, that a high accuracy is achieved.

KEYWORDS: Thin Kirchhoff Plates, Symmetric Galerkin Boundary Element Method

1 INTRODUCTION

Boundary integral formulations have been successfully developed and applied to the solution of plate bending problems since nearly thirty years. One of the first contributions is Jaswon and Maiti²⁵, and early references include Hansen²², Bezine⁷, Altiero and Sikarskie¹, Stern⁴². The book edited by Beskos⁵ provides a survey of the boundary element methods for plates and shells. Such formulations, based on the use of a fundamental solution for the biharmonic equation, consist of two coupled boundary integral equations, one strongly singular and the other hypersingular (i.e. containing singularities of order $1/r$ and $1/r^2$, respectively), associated with the representation formula of the flexural displacement and its normal derivative, respectively, at a point of the boundary. Numerical solutions are then sought for by collocating those integral equations at a finite number of boundary points. The singular boundary element integrals must be evaluated using either a direct method, Guiggiani¹⁹, or a regularization technique, Frangi¹⁴. Other contributions include Hartmann and Zotemantel²³, Brebbia, Telles and Wrobel¹³, Guoshu and Mukherjee²⁰ for static problems; Beskos⁶, Kitahara²⁷ for dynamic and vibration problems; Antes² for Reissner-Mindlin plates.

In all the above references, the collocation approach to boundary element discretization is used, leading to unsymmetric linear systems of equations. However, symmetric boundary integral equation formulations and their implementation have been investigated over the last twenty years. Early contributions include Nedelec³³ for potential problems, Sirtori³⁹ for static elasticity and Hamdi²¹ in acoustics. During the last few years, symmetric boundary integral equation formulations received an increasing attention in various areas of computational mechanics; see Sirtori *et al.*⁴⁰, Holzer²⁴, Kane and Balakrishna²⁶, Bonnet¹¹ for 2D and 3D elasticity; Gu and Yew¹⁸, Maier, Novati and Cen³⁰, Xu *et al.*⁴⁸ for fracture mechanics; Maier, Diligenti and Carini²⁸ for elastodynamics; Maier and Polizzotto³¹ and Maier *et al.*²⁹ for elastoplasticity, Pan, Maier and Amadei³⁵ for coupled problems such as poroelasticity. The symmetric boundary element formulations are surveyed by Bonnet, Maier and Polizzotto¹².

The purpose of the present paper is to derive symmetric boundary integral formulations for plate bending problems. Not much effort has been directed to this particular area of investigation. Tottenham⁴⁵ outlines how a symmetric formulation can be obtained by weighting in a Galerkin sense suitable equations containing different kind of sources: forces, moments, bi-couples and tri-couples. Singular integrations are next performed analytically under restrictive hypotheses on geometry- and field-modelling. More recently, Giroire and Nedelec¹⁷ and Nazaret³² derived Galerkin BIE formulations for plates with free edges, the presence of corners being allowed in the latter reference. In previous works for linear elasticity, several approaches have led to symmetric integral formulations. A first possibility consists in using static and kinematic sources (forces and displacement discontinuities) to generate an auxiliary elastic state which is exploited, together with the real field, in imposing the Betti theorem⁴⁰. A second possibility consists in taking weighted residuals of the displacement and traction boundary integral equations, using properly chosen test functions^{9, 26}; this is essentially a direct approach in the sense that the unknowns are the mechanical field quantities on the boundary. Both approaches can accommodate general boundary conditions; they are in fact equivalent in that they lead to the same final formulation. Finally, a third approach is based upon variational principles. Polizzotto³⁷ uses the Hu-Washizu principle⁴⁶, while Tereschenko⁴³ starts from the

stationarity of the elastic potential energy. The latter investigation is presented for Neumann boundary conditions only, but on the other hand also makes use of the complementary energy stationarity principle to formulate an *a posteriori* error estimation technique. Bonnet¹¹ uses the stationarity of the elastic potential energy, augmented with constraints expressing kinematical boundary conditions, so that Dirichlet, Neumann or mixed Dirichlet/Neumann are accommodated in the analysis; the direct symmetric boundary integral formulation thus obtained and the weighted residual formulation appear to be ultimately identical.

In the present paper the third, variational, approach will be extended to bending of linearly elastic Kirchhoff plates. Specifically, our symmetric Galerkin boundary element method (SGBEM) is obtained by exploiting the stationarity condition for the Lagrangian functional obtained by incorporating kinematical boundary conditions, in the form of constraint terms, into the elastic potential energy associated with bending when first-order variations of the unknown bending displacement solve the homogeneous elastic equilibrium equation. In the derivation of such a formulation one has to deal with very high potential singularities on the boundary, of order up to $1/r^4$. The evaluation of these terms is tackled using integration by parts, which is made possible thanks to the fact that the most singular kernels can be recast as derivatives of other, less singular, kernels with respect to the arc length along the boundary, extending an earlier work on a regularized collocation approach¹⁴. Our resulting formulation is direct, i.e. in terms of mechanical unknowns on the boundary (here, the bending displacement, its normal derivative, the normal moment and the Kirchhoff shear), and accommodates general boundary conditions. It involves singularities at most logarithmic, for which numerical quadrature techniques are available³⁶. It has been implemented using either straight or circular elements. Several numerical examples are presented, for various types of boundary conditions. Comparisons with analytical solutions show that accurate numerical results are obtained, even when the (potentially) highest kernel singularities are involved. It is important to stress that the three approaches outlined above are equivalent in the sense that the symmetric boundary integral formulations eventually obtained are identical. The variational viewpoint is thus preferred because starting from first principles provides, in the authors view, more insight. Although symmetric boundary integral formulations for plate problems with free edges appeared recently^{17, 32}, the present work is believed to be the first attempt at obtaining such formulations for general boundary conditions.

2 GEOMETRY AND GOVERNING EQUATIONS

Geometry. A thin undeformed plate of thickness h has its mid-surface S situated in the (x_1, x_2) plane in an orthonormal cartesian reference system (x_1, x_2, z) . Denote by $\mathbf{n}(\mathbf{x}) = [n_1, n_2](x_1, x_2)$ the unit normal to the boundary Γ of S pointing outwards of S and choose the unit tangent $\boldsymbol{\tau}(\mathbf{x}) = [\tau_1, \tau_2](x_1, x_2)$ to Γ so that $(\mathbf{n}, \boldsymbol{\tau}, \mathbf{e}_3)$ is a direct frame. The following relations will be used later:

$$\begin{aligned}\tau_i \tau_j + n_i n_j &= \delta_{ij} \\ n_i \tau_j - n_j \tau_i &= e_{ij} \\ \tau_j &= e_{ij} n_i \quad n_i = e_{ij} \tau_j\end{aligned}\tag{1}$$

Note that Γ needs not to be connected: if the surface S is multiply connected (i.e. the plate has holes), Γ is the union of an external boundary Γ_e and several internal boundaries.

Governing equations. The moment and shear components are defined from the three-dimensional stress tensor as

$$M_{ij}(\mathbf{x}) = \int_{-h/2}^{h/2} \sigma_{ij}(\mathbf{x}, z) z \, dz, \quad Q_i(\mathbf{x}) = \int_{-h/2}^{h/2} \sigma_{i3}(\mathbf{x}, z) \, dz, \quad (i, j) \in (1, 2)$$

Denote by $w(x_1, x_2)$ the out-of-plane bending displacement. In the linear elastic Kirchhoff theory, the moment and shear components, respectively denoted by M_{ij} and Q_i , are expressed in terms of w by

$$M_{ij} = -K_{ijkl} w_{,kl}, \quad Q_i = M_{ij,j} = -D w_{,ijj} \quad (2)$$

where

$$K_{ijkl} = D[(1 - \nu)\delta_{ik}\delta_{jl} + \nu\delta_{ij}\delta_{kl}], \quad D = \frac{Eh^3}{12(1 - \nu^2)} \quad (3)$$

(the comma indicates partial differentiation with respect to the field point and the Einstein summation convention is adopted for lower-case indices). E , ν and D are, respectively, the elastic modulus, the Poisson's coefficient and the flexural rigidity of the material. Moreover, the displacement w is governed by the elastic equilibrium equation

$$K_{ijkl} w_{,ijkl} = D w_{,iijj} = p(\mathbf{x}) \quad (4)$$

where $p(\mathbf{x})$ denotes the transverse load per unit area. The following general form of boundary conditions on Γ is considered:

$$\left\{ \begin{array}{ll} w = \bar{w} & \text{on } \Gamma_T \\ \varphi_N = \bar{\varphi}_N & \text{on } \Gamma_N \end{array} \right\}, \quad \left\{ \begin{array}{ll} M_N = \bar{M}_N & \text{on } \Gamma_M \\ Q_K = \bar{Q}_K & \text{on } \Gamma_Q \end{array} \right\} \quad (5)$$

where \bar{w} , $\bar{\varphi}_N$, \bar{M}_N , \bar{Q}_K denote given values for the displacement w , the normal slope $\varphi_N = w_{,i}n_i$, the normal moment $M_N = M_{ij}n_in_j$ and the Kirchhoff equivalent shear Q_K . Of course, the displacement data on Γ_T implies knowledge of the tangential slope:

$$\varphi_T = \bar{\varphi}_T = \frac{d\bar{w}}{ds} \quad \text{on } \Gamma_T$$

Besides, the Kirchhoff equivalent shear, defined as

$$Q_K = Q + \frac{dM_T}{ds} \quad (6)$$

(s : arc length along Γ) is a combination of the twisting moment $M_T = M_{ij}n_i\tau_j$ and the normal shear $Q = Q_in_i$, which are not independent of each other.

At any regular point of the boundary two such boundary conditions must be assigned. For any well-posed boundary value problem, $\Gamma_T \cap \Gamma_Q = \emptyset$, $\Gamma_T \cup \Gamma_Q = \Gamma$ and $\Gamma_N \cap \Gamma_M = \emptyset$, $\Gamma_N \cup \Gamma_M = \Gamma$ except, possibly, for transition regions at corner points (see Guoshu and Mukherjee²⁰ for an extensive study on the boundary conditions at corner points). At any corner point \mathbf{x}_c , a jump $\Delta_c M_T$ of the twisting moment is expected; moreover, either $w(\mathbf{x}_c)$ or $\Delta_c M_T$ is prescribed.

3 STATIONARITY OF AN AUGMENTED POTENTIAL ENERGY FUNCTIONAL

Using the foregoing definitions, the potential energy functional for a Kirchhoff plate reads

$$\mathcal{E}(w) = \frac{1}{2} \int_S w_{,kl} K_{ijkl} w_{,ij} dS + \int_{\Gamma_M} \bar{M}_N \varphi_N ds + \int_{\Gamma_Q} [\bar{M}_T \varphi_T - \bar{Q} w] ds - \int_S p w dS$$

where φ_T denotes the tangential derivative $\varphi_T = dw/ds = w_{,i} \tau_i$. Our goal is to establish a variational BEM formulation from the stationarity conditions of the following augmented potential energy functional:

$$\begin{aligned} \mathcal{F}(w, \lambda_N, \lambda_T, \lambda_Q) \\ = \mathcal{E}(w) + \int_{\Gamma_N} \lambda_N [\varphi_N - \bar{\varphi}_N] ds + \int_{\Gamma_T} \{\lambda_T [\varphi_T - \bar{\varphi}_T] - \lambda_Q [w - \bar{w}]\} ds \end{aligned}$$

where the kinematical boundary conditions on w , φ_T and φ_N in equation (5) appear as equality constraints with associated Lagrange multipliers $\lambda_N, \lambda_T, \lambda_Q$.

The first variation of the augmented potential energy functional \mathcal{F} is then given by

$$\begin{aligned} \delta \mathcal{F} = \int_S w_{,kl} K_{ijkl} \delta w_{,ij} dS + \int_{\Gamma_M} \bar{M}_N \delta \varphi_N ds + \int_{\Gamma_Q} [\bar{M}_T \delta \varphi_T - \bar{Q} \delta w] ds - \int_S p \delta w dS \\ + \int_{\Gamma_N} \lambda_N \delta \varphi_N ds + \int_{\Gamma_T} \{\lambda_T \delta \varphi_T - \lambda_Q \delta w\} ds \end{aligned} \quad (7)$$

It is important to note that, since the kinematical boundary conditions are incorporated into the Lagrangian $\delta \mathcal{F}$, the variations $\delta w, \delta \varphi_T, \delta \varphi_N$ in the above equation are not subjected to constraints. Setting $\delta \mathcal{F}$ equal to zero and exploiting the identity

$$\int_S M_{ij} \delta w_{,ij} dS = \int_{\Gamma} [M_{ij} n_j \delta w_{,i} - Q \delta w] ds + \int_S M_{ij,ij} \delta w dS \quad (8)$$

one obtains equation (4) and equation (5) and recognizes that the Lagrange multipliers λ_N , λ_T and λ_Q are respectively the normal moment on Γ_N , the twisting moment and the shear force on Γ_T , calculated from the displacement w .

Now let us restrict the first variation $\delta \mathcal{F}$ to variations δw which solve the homogeneous elastic equilibrium equation, that is

$$K_{ijkl} \delta w_{,ijkl} = 0 \quad (9)$$

For any such w one has

$$\int_S M_{ij} \delta w_{,ij} dS = \int_S \delta M_{ij} w_{,ij} dS = \int_{\Gamma} [\delta M_{ij} n_j w_{,i} - \delta Q w] ds \quad (10)$$

Imposing that $\delta \mathcal{F} = 0$ for any such δw and substituting the above identity into equation (7) leads to the following equality:

$$\int_{\Gamma} [Q \delta w - M_{ij} n_j \delta w_{,i} + w_{,i} \delta M_{ij} n_j - w \delta Q] ds + \int_S p \delta w dS = 0 \quad (11)$$

Note that, since the plate is in static equilibrium, one has

$$\int_{\Gamma} [Q(a + b_i x_i) - M_{ij} n_j b_i] ds + \int_S p(a + b_i x_i) dS = 0$$

where a and b_i are constants. Thus in (11) the kinematical test function δw ($\delta w_{,i}$ respectively) needs only to be defined up to a linear (constant) function of the coordinates.

Using $n_i n_j + \tau_i \tau_j = \delta_{ij}$, one has

$$M_{ij} n_j \delta w_{,i} = M_N \delta \varphi_N + M_T \delta \varphi_T$$

Then, an integration by parts of $M_T \delta \varphi_T$ gives

$$\int_{\Gamma} M_T \delta \varphi_T ds = - \sum_c \Delta_c M_T \delta w(\mathbf{x}_c) - \int_{\Gamma} \frac{d}{ds} M_T \delta w ds \quad (12)$$

where the \mathbf{x}_c are the (finitely many) corner points of Γ and $\Delta_c M_T$ is the jump of twisting moment at \mathbf{x}_c . Equation (11) thus admits the alternative form

$$\int_{\Gamma} [Q_K \delta w - M_N \delta \varphi_N + \delta M_{ij} n_j w_{,i} - \delta Q w] ds + \sum_c \Delta_c M_T \delta w(\mathbf{x}_c) + \int_S p \delta w dS = 0 \quad (13)$$

The stationarity equation (13) above is our starting point for building a Galerkin BIE formulation. Except for the last integral it involves boundary values of physical unknowns and test functions. This is a consequence of having considered only those test functions δw in elastic equilibrium, equation (9). It is worth noting at this point that eq. (13) is nothing else than the reciprocity theorem applied to any bending displacements w and δw that solve eqs. (4) and (9) respectively.

In order to put the stationarity equation (13) to actual use, it is now necessary to find a representation of all possible δw satisfying equation (9). This task relies on the use of integral representation formulas and is the subject of the next section.

4 TEST FUNCTIONS

4.1 Displacement test functions

Let \tilde{S} be a surface enclosed by a contour $\tilde{\Gamma}$ of unit outward normal $\tilde{\mathbf{n}}(\tilde{\mathbf{x}})$ and unit tangent $\tilde{\boldsymbol{\tau}}(\tilde{\mathbf{x}})$, such that $S \subset \tilde{S}$ strictly. The contour $\tilde{\Gamma}$ is defined by means of a one-to-one mapping onto Γ :

$$\mathbf{x} \in \Gamma \rightarrow \tilde{\mathbf{x}} = \mathbf{F}(\mathbf{x}; h) \quad (0 \leq h < h_0 \ll 1) \quad (14)$$

chosen such that for any corner point \mathbf{x}_c of Γ , $\tilde{\mathbf{x}}_c = \mathbf{F}(\mathbf{x}_c, h)$ is a corner point of $\tilde{\Gamma}$ and otherwise smooth. The mapping (14) depends on a small parameter h in such a way that

$$|\mathbf{F}(\mathbf{x}; h) - \mathbf{x}| \leq Ch \quad \mathbf{F}(\mathbf{x}; 0) = \mathbf{x} \quad (15)$$

According to these definitions, the exterior boundary Γ_e is surrounded by its counterpart $\tilde{\Gamma}_0$ whereas the internal boundaries $\Gamma - \Gamma_e$, if any, surround their counterpart $\tilde{\Gamma} - \tilde{\Gamma}_e$; this remark will have later relevance in some integration-by-parts manipulations.

Any δw defined on \tilde{S} and satisfying equation (9) may be represented by means of the integral representation (interior problem for \tilde{S}):

$$\begin{aligned}\delta w(\mathbf{x}) &= \int_{\tilde{\Gamma}} [W^*(\mathbf{x}, \tilde{\mathbf{x}}) \delta Q_K(\tilde{\mathbf{x}}) - \Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \delta M_N(\tilde{\mathbf{x}})] d\tilde{s} \\ &\quad + \int_{\tilde{\Gamma}} [M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \delta w_{,i}(\tilde{\mathbf{x}}) - Q^*(\mathbf{x}, \tilde{\mathbf{x}}) \delta w(\tilde{\mathbf{x}})] d\tilde{s} \\ &\quad + \sum_{\tilde{c}} W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) \delta \Delta_{\tilde{c}} M_T\end{aligned}\tag{16}$$

where the comma followed by a tilded lower-case letter indicates differentiation with respect to the corresponding $\tilde{\mathbf{x}}$ coordinate, \tilde{s} is the arc length defining the $\tilde{\mathbf{x}}$ point position. The kernel function, or fundamental solution, $W^*(\mathbf{x}, \tilde{\mathbf{x}})$, is any bending displacement generated at $\tilde{\mathbf{x}}$ by a unit point force acting at \mathbf{x} , i.e. any solution to equation (4) with $p(\tilde{\mathbf{x}}) = \delta(\mathbf{x} - \tilde{\mathbf{x}})$. One such solution is given by⁴²

$$W^*(\mathbf{x}, \tilde{\mathbf{x}}) = \frac{1}{16\pi D} r^2 \ln(r^2/r_0^2) \quad r = |\tilde{\mathbf{x}} - \mathbf{x}| \tag{17}$$

where r_0 is an arbitrary constant value. In the sequel, following Tottenham⁴⁵, r_0 is assumed such that $\ln r_0^2 = 1$. The normal slope, moment and shear associated with this fundamental solution are respectively given by

$$\Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) = W_{,i}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_i(\tilde{\mathbf{x}}) = \frac{1}{8\pi D} r \ln r^2 \frac{\partial r}{\partial \tilde{n}} \tag{18}$$

$$M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) = -K_{ijkl} W_{,k\tilde{l}}^*(\mathbf{x}, \tilde{\mathbf{x}}) = -\frac{1}{8\pi} \{2(1-\nu) r_{,i} r_{,\tilde{j}} + 2\nu \delta_{ij} + (1+\nu) \delta_{ij} \ln r^2\} \tag{19}$$

$$Q^*(\mathbf{x}, \tilde{\mathbf{x}}) = -D W_{,\tilde{j}\tilde{n}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_i(\tilde{\mathbf{x}}) = -\frac{1}{2\pi r} \frac{\partial r}{\partial \tilde{n}} \tag{20}$$

with

$$r_{,i} = \frac{\partial r}{\partial \tilde{x}_i} = \frac{\tilde{x}_i - x_i}{r} = -r_{,i} \quad \frac{\partial r}{\partial \tilde{n}} = r_{,i} \tilde{n}_i$$

Moreover, from equation (17), the following symmetry properties hold:

$$W^*(\mathbf{x}, \tilde{\mathbf{x}}) = W^*(\tilde{\mathbf{x}}, \mathbf{x}) \quad M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) = M_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \quad W_{,a}^*(\mathbf{x}, \tilde{\mathbf{x}}) = W_{,\tilde{a}}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tag{21}$$

Introduce the complementary surface $\tilde{S}^+ = \mathbf{R}^2 - \tilde{S}$; its boundary is again $\tilde{\Gamma}$ and, for consistency, the unit tangent and normal relative to \tilde{S}^+ are $\tilde{\boldsymbol{\tau}}^+ = -\tilde{\boldsymbol{\tau}}$ and $\tilde{\mathbf{n}}^+ = -\tilde{\mathbf{n}}$. Any δw^+ defined on \tilde{S}^+ , satisfying equation (9), verifies the exterior representation formula

$$\begin{aligned}0 &= \int_{\tilde{\Gamma}} [W^*(\mathbf{x}, \tilde{\mathbf{x}}) \delta Q_K^+(\tilde{\mathbf{x}}) + \Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \delta M_N^+(\tilde{\mathbf{x}})] d\tilde{s} \\ &\quad - \int_{\tilde{\Gamma}} [M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \delta w_{,i}^+(\tilde{\mathbf{x}}) - Q^*(\mathbf{x}, \tilde{\mathbf{x}}) \delta w^+(\tilde{\mathbf{x}})] d\tilde{s} \\ &\quad - \sum_{\tilde{c}} W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) \delta \Delta_{\tilde{c}} M_T^+(\tilde{\mathbf{x}}_c)\end{aligned}\tag{22}$$

Now, considering simultaneously an interior problem for the bounded plate \tilde{S} and an exterior problem for the unbounded plate \tilde{S}^+ having the same boundary data on $\tilde{\Gamma}$, adding equation (22) to equation (16) gives the most general representation for δw in \tilde{S} :

$$\begin{aligned}\delta w(\mathbf{x}) = & \int_{\tilde{\Gamma}} \left[W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) - \Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ & + \int_{\tilde{\Gamma}} \left[M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,i}(\tilde{\mathbf{x}}) - Q^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{w}(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ & + \sum_{\tilde{c}} W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) \Delta_{\tilde{c}} \tilde{M}_T\end{aligned}\quad (23)$$

where $\tilde{w} = \delta w - \delta w^+$, $\tilde{M}_N = \delta M_N - \delta M_N^+$, $\tilde{Q}_K = \delta Q_K + \delta Q_K^+$ and $\Delta_{\tilde{c}} \tilde{M}_T = \delta \Delta_{\tilde{c}} M_T - \delta \Delta_{\tilde{c}} M_T^+$. The jump of the cartesian derivatives $\tilde{w}_{,i} = \delta w_{,i} - \delta w_{,i}^+$ must be continuous along \tilde{S} . Moreover, the present definition of δw , δw^+ and so on implies that

$$\tilde{w} = 0 \quad \text{on } \tilde{\Gamma}_T, \quad \tilde{\varphi}_N = 0 \quad \text{on } \tilde{\Gamma}_N, \quad \tilde{M}_N = 0 \quad \text{on } \tilde{\Gamma}_M, \quad \tilde{Q}_K = 0 \quad \text{on } \tilde{\Gamma}_Q. \quad (24)$$

In addition, for a given corner point \mathbf{x}_c , $\tilde{w}(\mathbf{x}_c) = 0$ if $w(\mathbf{x}_c)$ is prescribed and $\Delta_c M_T$ unknown, $\Delta_c \tilde{M}_T = 0$ otherwise. Any test function δw of the form (23) solves the homogeneous elastic equilibrium equation, i.e. $K_{ijkl} \delta w_{,ijkl} = 0$. Moreover, from the above derivation, one readily sees that any sufficiently regular δw that solves $K_{ijkl} \delta w_{,ijkl} = 0$ admits a representation of the form equation (23).

In line with a previous work¹¹ for three-dimensional elasticity, the limit $\tilde{\Gamma} \rightarrow \Gamma$, i.e. $h \rightarrow 0$, will be taken in the above definition of the test function δw together with its derived quantities $\delta \varphi_N$, δM_{ij} , δQ ; the resulting expressions will then be substituted in the stationarity condition equation (13). The resulting equation must hold true for every $\tilde{\varphi}_T$, $\tilde{\varphi}_N$, \tilde{M}_N , \tilde{Q}_K , $\Delta_{\tilde{c}} \tilde{M}_T$ under the constraints (24). Hence, following the usual variational calculus argument, five independent equations will be obtained by considering first the case $\tilde{\varphi}_T \neq 0$, $\tilde{\varphi}_N = \tilde{M}_N = \tilde{Q}_K = \Delta_{\tilde{c}} \tilde{M}_T = 0$, next $\tilde{\varphi}_T = 0$, $\tilde{\varphi}_N \neq 0$, $\tilde{M}_N = \tilde{Q}_K = \Delta_{\tilde{c}} \tilde{M}_T = 0$, and so on.

First, multiply equation (23) by $p(\mathbf{x})$ and integrate the result over S . The most singular kernel, $Q^*(\mathbf{x}, \tilde{\mathbf{x}})$, is integrable over S , so that the limiting process $h \rightarrow 0$ can be performed at once and simply yields:

$$\begin{aligned}\int_S p(\mathbf{x}) \delta w(\mathbf{x}) dS = & \int_S p(\mathbf{x}) \int_{\Gamma} \left[W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) - \Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} dS \\ & + \int_S p(\mathbf{x}) \int_{\Gamma} \left[M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,i}(\tilde{\mathbf{x}}) - Q^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{w}(\tilde{\mathbf{x}}) \right] d\tilde{s} dS \\ & + \sum_{\tilde{c}} \int_S p(\mathbf{x}) W^*(\mathbf{x}, \mathbf{x}_c) \Delta_{\tilde{c}} \tilde{M}_T dS\end{aligned}\quad (25)$$

Moreover, as shown in Appendix B, the domain integrals in the above formula can be reformulated as boundary integrals for some particular classes of loading p , e.g. when p is a harmonic function (this includes of course the case of uniform p).

In contrast, the singularity of the kernel $Q^*(\mathbf{x}, \tilde{\mathbf{x}})$ is such that a single curvilinear integral becomes divergent in the limit $h \rightarrow 0$, which thus cannot be directly taken in equation (23). However, as shown in Appendix A.1, one has

$$Q^*(\mathbf{x}, \tilde{\mathbf{x}}) = \frac{d}{d\tilde{s}} R^*(\mathbf{x}, \tilde{\mathbf{x}}), \quad \text{with} \quad R^*(\mathbf{x}, \tilde{\mathbf{x}}) = -\frac{1}{2\pi} \theta(\mathbf{x}, \tilde{\mathbf{x}}) \quad (26)$$

where $\theta(\mathbf{x}, \tilde{\mathbf{x}}) = (\widehat{\mathbf{e}, \mathbf{r}})$ is the angle between an arbitrarily chosen reference direction \mathbf{e} and the position vector $\mathbf{r} = \tilde{\mathbf{x}} - \mathbf{x}$. Then the strongly singular term in (23) can be integrated by parts. To this end, it is important to note that the function $\tilde{\mathbf{x}} \rightarrow \theta(\mathbf{x}, \tilde{\mathbf{x}})$ presents a jump of magnitude 2π across an arbitrarily chosen point $\tilde{\mathbf{x}}_0 \in \tilde{\Gamma}_e$, whereas it is continuous for $\tilde{\mathbf{x}} \in \tilde{\Gamma} - \tilde{\Gamma}_e$, since \mathbf{x} is interior to $\tilde{\Gamma}_e$ but exterior to any component of $\tilde{\Gamma} - \tilde{\Gamma}_e$; this distinction will be materialized by the use of a function κ , defined on the plate boundary by:

$$\kappa(\mathbf{x}) = 1 \quad (\mathbf{x} \in \Gamma_e) \quad \kappa(\mathbf{x}) = 0 \quad (\mathbf{x} \in \Gamma - \Gamma_e) \quad (27)$$

Moreover, in anticipation of the eventual limit process $h \rightarrow 0$, the particular choice $\tilde{\mathbf{x}}_0 = \mathbf{F}(\mathbf{x}, h)$ is made.

The potentially strongly singular term in (23) is now integrated by parts:

$$\begin{aligned} \int_{\tilde{\Gamma}} Q^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{w}(\tilde{\mathbf{x}}) d\tilde{s} &= -\frac{1}{2\pi} [\theta(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{w}(\tilde{\mathbf{x}})]_{\tilde{\mathbf{x}}_0^+}^{\tilde{\mathbf{x}}_0^-} + e_{ij} \int_{\tilde{\Gamma}} R^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,i}(\tilde{\mathbf{x}}) d\tilde{s} \\ &= -\tilde{w}(\tilde{\mathbf{x}}_0) \kappa(\tilde{\mathbf{x}}_0) + e_{ij} \int_{\tilde{\Gamma}} R^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,i}(\tilde{\mathbf{x}}) d\tilde{s} \end{aligned} \quad (28)$$

Then, equation (23) becomes:

$$\begin{aligned} \delta w(\mathbf{x}) &= \int_{\tilde{\Gamma}} \left[W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) - \Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ &\quad + \int_{\tilde{\Gamma}} P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,i}(\tilde{\mathbf{x}}) d\tilde{s} + \sum_{\tilde{c}} W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) \Delta_{\tilde{c}} \tilde{M}_T + \tilde{w}(\tilde{\mathbf{x}}_0) \kappa(\tilde{\mathbf{x}}_0) \end{aligned} \quad (29)$$

using the auxiliary, weakly singular, kernel $P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) = M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) - e_{ij} R^*(\mathbf{x}, \tilde{\mathbf{x}})$.

At this point, one notes that all integrals in (29) are at most weakly singular, so that the same identity holds, with $\tilde{\Gamma}$ replaced by Γ (and hence $\tilde{\boldsymbol{\tau}}, \tilde{\mathbf{n}}$ replaced by $\boldsymbol{\tau}, \mathbf{n}$ as well), for the limiting case $\tilde{\Gamma} \rightarrow \Gamma$. Upon multiplication of the resulting identity by $Q_K(\mathbf{x})$ and integration for $\mathbf{x} \in \Gamma$ and recalling that $\tilde{\mathbf{x}}_0 = \mathbf{F}(\mathbf{x}, h)$, one finally gets:

$$\begin{aligned} \int_{\Gamma} Q_K(\mathbf{x}) \delta w(\mathbf{x}) d\mathbf{s} &= \int_{\Gamma} \int_{\Gamma} Q_K(\mathbf{x}) \left[W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) - \Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} d\mathbf{s} \\ &\quad + \int_{\Gamma} \int_{\Gamma} Q_K(\mathbf{x}) P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,i}(\tilde{\mathbf{x}}) d\tilde{s} d\mathbf{s} \\ &\quad + \int_{\Gamma} Q_K(\mathbf{x}) \tilde{w}(\mathbf{x}) \kappa(\mathbf{x}) d\mathbf{s} + \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_{\Gamma} Q_K(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) d\mathbf{s} \end{aligned} \quad (30)$$

Similarly, equation (29) with $\mathbf{x} = \mathbf{x}_c$ gives

$$\begin{aligned} \sum_c \delta w(\mathbf{x}_c) \Delta_c M_T &= \sum_c \Delta_c M_T \int_{\Gamma} \left[W^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) - \Phi_N^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ &\quad + \sum_c \Delta_c M_T \int_{\Gamma} P_{ij}^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,i}(\tilde{\mathbf{x}}) d\tilde{s} \\ &\quad + \sum_c \sum_{\tilde{c}} \Delta_c M_T W^*(\mathbf{x}_c, \tilde{\mathbf{x}}_c) \Delta_{\tilde{c}} \tilde{M}_T + \sum_c \tilde{w}(\mathbf{x}_c) \Delta_c M_T \kappa(\mathbf{x}_c) \end{aligned} \quad (31)$$

Note that the kernel $R^*(\mathbf{x}, \tilde{\mathbf{x}})$ is defined up to an additive constant (which value is irrelevant) and, besides, is necessarily such that, for $\mathbf{x}, \tilde{\mathbf{x}} \in \Gamma_e$:

$$R(\mathbf{x}, \mathbf{x}^+) - R(\mathbf{x}, \mathbf{x}^-) = -1$$

i.e. for a fixed $\mathbf{x} \in \Gamma_e$, the 2π angle jump on Γ_e must occur precisely at $\tilde{\mathbf{x}} = \mathbf{x}$.

4.2 Gradient of displacement test functions

Let us now differentiate equation (29) with respect to x_k :

$$\begin{aligned} \delta w_{,k}(\mathbf{x}) = & \int_{\tilde{\Gamma}} \left[W_{,k}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) - W_{,k\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_i(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ & + \int_{\tilde{\Gamma}} P_{ij,k}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) \tilde{w}_{,\tilde{i}}(\tilde{\mathbf{x}}) d\tilde{s} + \sum_{\tilde{c}} W_{,k}^*(\mathbf{x}, \tilde{\mathbf{x}}_c) \Delta_{\tilde{c}} \tilde{M}_T \end{aligned} \quad (32)$$

As shown by Frangi¹⁴, the kernel derivative $W_{,k}^*$ can be interpreted as the displacement at $\tilde{\mathbf{x}}$ caused by a unit concentrated moment applied at \mathbf{x} in the k -th direction.

In the above equation, the kernel $P_{ij,k}^*(\mathbf{x}, \tilde{\mathbf{x}})$ is potentially strongly singular. However, as shown in Appendix A.2, the following identity holds:

$$P_{ij,k}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) = -P_{ij,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) = -e_{jk} \frac{d}{d\tilde{s}} P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \quad (33)$$

so that the integral containing $P_{ij,k}^*(\mathbf{x}, \tilde{\mathbf{x}})$ can be integrated by parts. Following the same steps as for equation (28), equation (32) becomes

$$\begin{aligned} \delta w_{,k}(\mathbf{x}) = & \int_{\tilde{\Gamma}} \left[W_{,k}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) - W_{,k\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_i(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ & - \int_{\tilde{\Gamma}} e_{kj} P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \frac{d}{d\tilde{s}} \tilde{w}_{,\tilde{i}}(\tilde{\mathbf{x}}) d\tilde{s} + \sum_{\tilde{c}} W_{,k}^*(\mathbf{x}, \tilde{\mathbf{x}}_c) \Delta_{\tilde{c}} \tilde{M}_T + \tilde{w}_{,k}(\tilde{\mathbf{x}}_0) \kappa(\tilde{\mathbf{x}}_0) \end{aligned} \quad (34)$$

At this stage, all integrals in (34) are again at most weakly singular, so that the same identity holds, with $\tilde{\Gamma}$ replaced by Γ , for the limiting case $\tilde{\Gamma} \rightarrow \Gamma$. Finally, upon multiplication of the resulting identity by $M_N(\mathbf{x}) n_k(\mathbf{x})$ and integration for $\mathbf{x} \in \Gamma$, one gets

$$\begin{aligned} \int_{\Gamma} M_N(\mathbf{x}) \delta \varphi_N(\mathbf{x}) ds = & \int_{\Gamma} \int_{\Gamma} M_N(\mathbf{x}) \Phi_N^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\ & - \int_{\Gamma} \int_{\Gamma} M_N(\mathbf{x}) W_{,k\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_k(\mathbf{x}) n_i(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\ & - \int_{\Gamma} \int_{\Gamma} M_N(\mathbf{x}) P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tau_j(\mathbf{x}) \frac{d}{d\tilde{s}} \tilde{w}_{,\tilde{i}}(\tilde{\mathbf{x}}) d\tilde{s} ds \\ & + \int_{\Gamma} M_N(\mathbf{x}) \tilde{\varphi}_N(\mathbf{x}) \kappa(\mathbf{x}) ds + \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_{\Gamma} M_N(\mathbf{x}) \Phi_N^*(\tilde{\mathbf{x}}_c, \mathbf{x}) ds \end{aligned} \quad (35)$$

4.3 Moment and shear test functions

The moment δM_{ij} and shear δQ associated to δw through equations (2) are obtained by means of eqs. (19,20); they are given by:

$$\begin{aligned}\delta M_{ij}(\mathbf{x}) &= \int_{\tilde{\Gamma}} \left[M_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{Q}_K(\tilde{\mathbf{x}}) - M_{ij,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{n}_k(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ &\quad - \int_{\tilde{\Gamma}} K_{ijab} P_{k\ell,ab}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_\ell(\tilde{\mathbf{x}}) \tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} + \sum_{\tilde{c}} M_{ij}^*(\tilde{\mathbf{x}}_c, \mathbf{x}) \Delta_{\tilde{c}} \tilde{M}_T \\ \delta Q(\mathbf{x}) &= \int_{\tilde{\Gamma}} \left[Q^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{Q}_K(\tilde{\mathbf{x}}) - Q_{,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{n}_k(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} \\ &\quad - n_i(\mathbf{x}) \int_{\tilde{\Gamma}} K_{ijab} P_{k\ell,abj}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_\ell(\tilde{\mathbf{x}}) \tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} + \sum_{\tilde{c}} Q^*(\tilde{\mathbf{x}}_c, \mathbf{x}) \Delta_{\tilde{c}} \tilde{M}_T\end{aligned}$$

The limiting expression when $h \rightarrow 0$ of the quantity

$$\int_{\Gamma} (\delta M_{ij}(\mathbf{x}) w_{,i}(\mathbf{x}) n_j(\mathbf{x}) - \delta Q(\mathbf{x}) w(\mathbf{x})) ds \quad (36)$$

is now sought. This task necessitates, again, some integrations by parts. First, using again equation (26) and noting that the function $\mathbf{x} \rightarrow \theta(\mathbf{x}, \tilde{\mathbf{x}})$ is continuous over the external boundary Γ_e and has a -2π -jump over $\Gamma - \Gamma_e$ at some point \mathbf{x}_0 (the minus sign of the jump stems from consistency of orientation conventions for the external and internal boundary curves), one has:

$$\int_{\Gamma} w(\mathbf{x}) Q^*(\tilde{\mathbf{x}}, \mathbf{x}) ds = w(\mathbf{x}_0)(1 - \kappa(\mathbf{x}_0)) + e_{ij} \int_{\Gamma} n_j(\mathbf{x}) w_{,i}(\mathbf{x}) R^*(\tilde{\mathbf{x}}, \mathbf{x}) ds \quad (37)$$

$$\int_{\Gamma} w(\mathbf{x}) Q_{,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) ds = e_{ij} \int_{\Gamma} n_j(\mathbf{x}) w_{,i}(\mathbf{x}) R_{,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) ds \quad (38)$$

(using $\tau_i = -e_{ij} n_j$). Next, using equation (38), one has

$$\begin{aligned}\int_{\Gamma} \int_{\tilde{\Gamma}} \left\{ w_{,i}(\mathbf{x}) n_j(\mathbf{x}) M_{ij,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) - w(\mathbf{x}) Q_{,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) \right\} \tilde{n}_k(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\ = \int_{\Gamma} \int_{\tilde{\Gamma}} w_{,i}(\mathbf{x}) n_j(\mathbf{x}) P_{ij,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{n}_k(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds\end{aligned} \quad (39)$$

Upon noting that, similarly to equation (33), one has

$$P_{ij,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) n_j(\mathbf{x}) \tilde{n}_k(\tilde{\mathbf{x}}) = -e_{jk} \frac{d}{ds} P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{n}_k(\tilde{\mathbf{x}}) = \tilde{\tau}_j(\tilde{\mathbf{x}}) \frac{d}{ds} P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x})$$

and that the cartesian derivatives $w_{,i}$ are continuous throughout Γ , the following integration by parts applies:

$$\begin{aligned}\int_{\Gamma} \int_{\tilde{\Gamma}} w_{,i}(\mathbf{x}) n_j(\mathbf{x}) P_{ij,\tilde{k}}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{n}_k(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\ = - \int_{\Gamma} \int_{\tilde{\Gamma}} \frac{d}{ds} w_{,i}(\mathbf{x}) P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{\tau}_j(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds - \int_{\Gamma} \varphi_N(\mathbf{x}) \tilde{M}_N(\mathbf{x}_0) (1 - \kappa(\mathbf{x}_0)) ds\end{aligned} \quad (40)$$

Then, from the identity:

$$K_{ijab}P_{k\ell,abj}^*(\mathbf{x}, \tilde{\mathbf{x}})n_i(\mathbf{x}) = K_{k\ell ab}\frac{d}{ds}R_{,ab}^*(\mathbf{x}, \tilde{\mathbf{x}}) \quad (41)$$

which is established in Appendix A.3, one has

$$\begin{aligned} \int_{\Gamma} w(\mathbf{x})n_i(\mathbf{x}) \int_{\tilde{\Gamma}} K_{ijab}P_{k\ell,abj}^*(\mathbf{x}, \tilde{\mathbf{x}})\tilde{n}_{\ell}(\tilde{\mathbf{x}})\tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} ds \\ = e_{ij} \int_{\Gamma} w_{,i}(\mathbf{x})n_j(\mathbf{x}) \int_{\tilde{\Gamma}} K_{k\ell ab}R_{,ab}^*(\mathbf{x}, \tilde{\mathbf{x}})\tilde{n}_{\ell}(\tilde{\mathbf{x}})\tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} ds \end{aligned} \quad (42)$$

(note that $R_{,ab}^*(\mathbf{x}, \tilde{\mathbf{x}})$ is continuous for $\mathbf{x} \in \Gamma$). Besides, it is shown in Appendix A.4 that the tensor $Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}})$ given by

$$Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) = -D^2 \left[(1 - \nu^2)W_{,a\tilde{a}}^*(\mathbf{x}, \tilde{\mathbf{x}})\delta_{ik} + (1 - \nu)^2W_{,\tilde{i}\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \right]$$

is such that

$$\left[e_{ij}K_{k\ell ab}R_{,ab}^*(\mathbf{x}, \tilde{\mathbf{x}}) - K_{ijab}P_{k\ell,ab}^*(\mathbf{x}, \tilde{\mathbf{x}}) \right] n_j(\mathbf{x})\tilde{n}_{\ell}(\tilde{\mathbf{x}}) = \frac{d}{ds}\frac{d}{d\tilde{s}}Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) \quad (43)$$

Note that this kernel has also been found, independently and recently, by Giroire and Nedelec¹⁷. Combining eqs. (42) and (43), one obtains:

$$\begin{aligned} \int_{\Gamma} \int_{\tilde{\Gamma}} \left\{ w_{,i}(\mathbf{x})n_j(\mathbf{x})K_{ijab}P_{k\ell,ab}^*(\mathbf{x}, \tilde{\mathbf{x}}) - w(\mathbf{x})K_{ijab}P_{k\ell,abj}^*(\mathbf{x}, \tilde{\mathbf{x}})n_i(\mathbf{x}) \right\} \tilde{n}_{\ell}(\tilde{\mathbf{x}})\tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} ds \\ = - \int_{\Gamma} \int_{\tilde{\Gamma}} w_{,i}(\mathbf{x})\frac{d}{ds}\frac{d}{d\tilde{s}}Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}})\tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} ds \\ = - \int_{\Gamma} \int_{\tilde{\Gamma}} \frac{d}{ds}w_{,i}(\mathbf{x})Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}})\frac{d}{d\tilde{s}}\tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} ds \end{aligned} \quad (44)$$

The kernels in eqs. (37), (38), (40), (44) are at most weakly singular. Following the now usual argument, the limiting expression for $\tilde{\Gamma} \rightarrow \Gamma$ of equation (36) is:

$$\begin{aligned} \int_{\Gamma} (\delta M_{ij}(\mathbf{x})w_{,i}(\mathbf{x})n_j(\mathbf{x}) - \delta Q(\mathbf{x})w(\mathbf{x})) ds \\ = \int_{\Gamma} \int_{\Gamma} \left[w_{,i}(\mathbf{x})n_j(\mathbf{x})P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x})\tilde{Q}_K(\tilde{\mathbf{x}}) + \frac{d}{ds}w_{,i}(\mathbf{x})\tau_j(\tilde{\mathbf{x}})P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x})\tilde{M}_N(\tilde{\mathbf{x}}) \right] d\tilde{s} ds \\ + \int_{\Gamma} \int_{\Gamma} \frac{d}{ds}w_{,i}(\mathbf{x})Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}})\frac{d}{d\tilde{s}}\tilde{w}_{,\tilde{k}}(\tilde{\mathbf{x}}) d\tilde{s} ds \\ - \int_{\Gamma} w(\mathbf{x})\tilde{Q}_K(\mathbf{x})(1 - \kappa(\mathbf{x})) ds + \int_{\Gamma} \varphi_N(\mathbf{x})\tilde{M}_N(\mathbf{x})(1 - \kappa(\mathbf{x})) ds \\ + \sum_{\tilde{c}} \Delta_{\tilde{c}}\tilde{M}_T \int_{\Gamma} w_{,i}(\mathbf{x})P_{ij}^*(\tilde{\mathbf{x}}_c, \mathbf{x})n_j(\mathbf{x}) ds - \sum_{\tilde{c}} \Delta_{\tilde{c}}\tilde{M}_T w(\tilde{\mathbf{x}}_c)(1 - \kappa(\tilde{\mathbf{x}}_c)) \end{aligned} \quad (45)$$

5 THE VARIATIONAL FORMULATION

At this point, the sought-for variational boundary integral formulation is set up by means of the following steps:

1. Equations (25), (30), (31), (35), (45) are substituted into the stationarity equation (13);
2. Every occurrence of $w_{,i}$ and $\tilde{w}_{,i}$ is replaced by $n_i\varphi_N + \tau_i\varphi_T$ and $n_i\tilde{\varphi}_N + \tau_i\tilde{\varphi}_T$ respectively;
3. The boundary condition structure is explicitly incorporated, i.e. all integrals are split over the appropriate boundary subsets $\Gamma_T, \Gamma_N, \Gamma_M, \Gamma_Q$ of Γ , the boundary data $\tilde{\varphi}_T, \tilde{\varphi}_N, \tilde{M}_N, \tilde{Q}_K$ (and hence the true unknowns) are made to appear explicitly, the constraints (24) on the test functions on Γ being taken into account. Similarly, the summations over corners are split according to whether ΔM_T or w is prescribed.

Those numerous but straightforward manipulations result in the final form of the stationarity equation (13). It has the following general structure:

$$\forall(\tilde{\varphi}_T, \tilde{\varphi}_N, \tilde{M}_N, \tilde{Q}_K, \Delta\tilde{M}_T(\mathbf{x}_c))$$

$$\int_{\Gamma_Q} \mathcal{G}_Q \tilde{\varphi}_T ds + \int_{\Gamma_M} \mathcal{G}_M \tilde{\varphi}_N ds + \int_{\Gamma_N} \mathcal{G}_N \tilde{M}_N ds + \int_{\Gamma_T} \mathcal{G}_T \tilde{Q}_K ds + \sum_c \mathcal{G}_\Delta \Delta\tilde{M}_T(\mathbf{x}_c) = 0$$

where $\mathcal{G}_Q, \mathcal{G}_M, \mathcal{G}_N, \mathcal{G}_T, \mathcal{G}_\Delta$ are integral operators. According to the usual argument of the calculus of variations, this imply that each of the five terms of the above sum should vanish separately. The stationarity equation (13) thus leads to the following system of equations:

$$(\forall \tilde{Q}_K, \tilde{M}_N, \tilde{\varphi}_N, \tilde{\varphi}_T, \Delta\tilde{M}_T)$$

$$\begin{pmatrix} \tilde{Q}_K \\ \tilde{M}_N \\ \tilde{\varphi}_N \\ \tilde{\varphi}_T \\ \Delta\tilde{M}_T \end{pmatrix}^T \begin{bmatrix} \mathcal{B}_{QQ} & \mathcal{B}_{QM} & \mathcal{B}_{QN} & \mathcal{B}_{QT} & \mathcal{B}_{Q\Delta} \\ \mathcal{B}_{MQ} & \mathcal{B}_{MM} & \mathcal{B}_{MN} & \mathcal{B}_{MT} & \mathcal{B}_{M\Delta} \\ \mathcal{B}_{NQ} & \mathcal{B}_{NM} & \mathcal{B}_{NN} & \mathcal{B}_{NT} & \mathcal{B}_{N\Delta} \\ \mathcal{B}_{TQ} & \mathcal{B}_{TM} & \mathcal{B}_{TN} & \mathcal{B}_{TT} & \mathcal{B}_{T\Delta} \\ \mathcal{B}_{\Delta Q} & \mathcal{B}_{\Delta M} & \mathcal{B}_{\Delta N} & \mathcal{B}_{\Delta T} & \mathcal{B}_{\Delta\Delta} \end{bmatrix} \begin{pmatrix} Q_K \\ M_N \\ \varphi_N \\ \varphi_T \\ \Delta M_T \end{pmatrix} = \begin{pmatrix} \tilde{Q}_K \\ \tilde{M}_N \\ \tilde{\varphi}_N \\ \tilde{\varphi}_T \\ \Delta\tilde{M}_T \end{pmatrix}^T \begin{pmatrix} \mathcal{L}_Q \\ \mathcal{L}_M \\ \mathcal{L}_N \\ \mathcal{L}_T \\ \mathcal{L}_\Delta \end{pmatrix} \quad (46)$$

where the bilinear forms $f^T \mathcal{B}_{XY} g \equiv \mathcal{B}_{XY}(f, g)$ (with $(X, Y) \in \{Q, M, N, T, \Delta\}$) are given as follows:

$$\begin{aligned} \mathcal{B}_{QQ}(Q_K, \tilde{Q}_K) &= \int_{\Gamma_T} \int_{\Gamma_T} Q_K(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\ \mathcal{B}_{QM}(M_N, \tilde{Q}_K) &= - \int_{\Gamma_N} \int_{\Gamma_T} M_N(\mathbf{x}) \Phi_N^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\ \mathcal{B}_{QN}(\varphi_N, \tilde{Q}_K) &= \int_{\Gamma_M} \int_{\Gamma_T} \varphi_N(\mathbf{x}) M_N^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\ \mathcal{B}_{QT}(\varphi_T, \tilde{Q}_K) &= \int_{\Gamma_Q} \int_{\Gamma_T} \varphi_T(\mathbf{x}) [R^*(\tilde{\mathbf{x}}, \mathbf{x}) + M_T^*(\tilde{\mathbf{x}}, \mathbf{x})] \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\ \mathcal{B}_{Q\Delta}(\Delta M_T, \tilde{Q}_K) &= \sum_c \Delta_c M_T \int_{\Gamma_T} W^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} \end{aligned} \quad (47)$$

$$\begin{aligned}
\mathcal{B}_{MM}(M_N, \tilde{M}_N) &= \int_{\Gamma_N} \int_{\Gamma_N} M_N(\mathbf{x}) W_{,k\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_k(\mathbf{x}) n_i(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
\mathcal{B}_{MN}(\varphi_N, \tilde{M}_N) &= \int_{\Gamma_M} \int_{\Gamma_N} D_i^N \varphi_N(\mathbf{x}) P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tau_j(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
\mathcal{B}_{MT}(\varphi_T, \tilde{M}_N) &= \int_{\Gamma_Q} \int_{\Gamma_N} D_i^T \varphi_T(\mathbf{x}) P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tau_j(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
\mathcal{B}_{M\Delta}(\Delta M_T, \tilde{M}_N) &= - \sum_c \Delta_c M_T \int_{\Gamma_N} \Phi_N^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} \quad (48)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{NN}(\varphi_N, \tilde{\varphi}_N) &= \int_{\Gamma_M} \int_{\Gamma_M} D_i^N \varphi_N(\mathbf{x}) Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) D_k^N \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
\mathcal{B}_{NT}(\varphi_T, \tilde{\varphi}_N) &= \int_{\Gamma_Q} \int_{\Gamma_M} D_i^T \varphi_T(\mathbf{x}) Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) D_k^N \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
\mathcal{B}_{N\Delta}(\Delta M_T, \tilde{\varphi}_N) &= \sum_c \Delta_c M_T \int_{\Gamma_M} M_N^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} \quad (49)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{TT}(\varphi_T, \tilde{\varphi}_T) &= \int_{\Gamma_Q} \int_{\Gamma_Q} D_i^T \varphi_T(\mathbf{x}) Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) D_k^T \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} ds \\
\mathcal{B}_{T\Delta}(\Delta M_T, \tilde{\varphi}_T) &= - \sum_c \Delta_c M_T \int_{\Gamma_Q} [M_T^*(\mathbf{x}_c, \tilde{\mathbf{x}}) + R^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{\varphi}_T(\tilde{\mathbf{x}})] d\tilde{s} \quad (50)
\end{aligned}$$

$$\mathcal{B}_{\Delta\Delta}(\Delta M_T, \Delta \tilde{M}_T) = \sum_c \Delta_c M_T \sum_{\tilde{c}} W^*(\mathbf{x}_c, \tilde{\mathbf{x}}_c) \Delta_{\tilde{c}} \tilde{M}_T \quad (51)$$

and the linear forms $f^T \mathcal{L}_X \equiv \mathcal{L}_X(f)$ as follows:

$$\begin{aligned}
\mathcal{L}_Q(\tilde{Q}_K) &= - \int_{\Gamma_Q} \int_{\Gamma_T} \bar{Q}_K(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\
&\quad + \int_{\Gamma_M} \int_{\Gamma_T} \bar{M}_N(\mathbf{x}) \Phi_N^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\
&\quad - \int_{\Gamma_N} \int_{\Gamma_T} \bar{\varphi}_N(\mathbf{x}) M_N^*(\tilde{\mathbf{x}}, \mathbf{x}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\
&\quad - \int_{\Gamma_T} \int_{\Gamma_T} \bar{\varphi}_T(\mathbf{x}) [M_T^*(\tilde{\mathbf{x}}, \mathbf{x}) + R^*(\tilde{\mathbf{x}}, \mathbf{x})] \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} ds \\
&\quad + \int_{\Gamma_T} \bar{w}(\mathbf{x}) \tilde{Q}_K(\mathbf{x}) (1 - \kappa(\mathbf{x})) ds \\
&\quad - \int_S \int_{\Gamma_T} p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} dS \\
&\quad - \sum_d \overline{\Delta_d M_T} \int_{\Gamma_T} W^*(\mathbf{x}_c, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} \quad (52)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_M(\tilde{M}_N) = & \int_{\Gamma_Q} \int_{\Gamma_N} \bar{Q}_K(\mathbf{x}) W_{,i}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_i(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_{\Gamma_M} \int_{\Gamma_N} \bar{M}_N(\mathbf{x}) W_{,ki}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_k(\mathbf{x}) n_i(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_{\Gamma_N} \int_{\Gamma_N} D_i^N \bar{\varphi}_N(\mathbf{x}) P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tau_j(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_{\Gamma_N} \bar{\varphi}_N(\mathbf{x}) \tilde{M}_N(\mathbf{x}) (1 - \kappa(\mathbf{x})) ds \\
& - \int_{\Gamma_T} \int_{\Gamma_N} D_i^T \bar{\varphi}_T(\mathbf{x}) P_{ij}^*(\tilde{\mathbf{x}}, \mathbf{x}) \tau_j(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& + \int_S \int_{\Gamma_N} p(\mathbf{x}) \Phi_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} dS \\
& + \sum_d \overline{\Delta_d M_T} \int_{\Gamma_N} \Phi_N^*(\mathbf{x}_d, \tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s}
\end{aligned} \tag{53}$$

$$\begin{aligned}
\mathcal{L}_N(\tilde{\varphi}_N) = & - \int_{\Gamma_Q} \int_{\Gamma_M} \bar{Q}_K(\mathbf{x}) \Phi_N^*(\tilde{\mathbf{x}}, \mathbf{x}) d\tilde{s} ds \\
& - \int_{\Gamma_M} \int_{\Gamma_M} \bar{M}_N(\mathbf{x}) P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tau_j(\mathbf{x}) D_i^N \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& + \int_{\Gamma_M} \bar{M}_N(\mathbf{x}) \tilde{\varphi}_N(\mathbf{x}) \kappa(\mathbf{x}) ds \\
& - \int_{\Gamma_N} \int_{\Gamma_M} D_i^N \bar{\varphi}_N(\mathbf{x}) Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) D_i^N \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_{\Gamma_T} \int_{\Gamma_M} D_i^T \bar{\varphi}_T(\mathbf{x}) Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) D_i^N \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_S \int_{\Gamma_M} p(\mathbf{x}) M_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} dS \\
& - \sum_d \overline{\Delta_d M_T} \int_{\Gamma_M} M_N^*(\mathbf{x}_d, \tilde{\mathbf{x}}) \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s}
\end{aligned} \tag{54}$$

$$\begin{aligned}
\mathcal{L}_T(\tilde{\varphi}_T) = & - \int_{\Gamma_Q} \int_{\Gamma_Q} \bar{Q}_K(\mathbf{x}) [M_T^*(\mathbf{x}, \tilde{\mathbf{x}}) + R^*(\mathbf{x}, \tilde{\mathbf{x}})] \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_{\Gamma_Q} \bar{Q}_K(\mathbf{x}) \tilde{w}(\mathbf{x}) \kappa(\mathbf{x}) ds - \sum_d \tilde{w}(\mathbf{x}_c) \overline{\Delta_c M_T} \kappa(\mathbf{x}_c) \\
& - \int_{\Gamma_M} \int_{\Gamma_Q} \bar{M}_N(\mathbf{x}) P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tau_j(\mathbf{x}) D_i^T \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_{\Gamma_N} \int_{\Gamma_Q} D_i^N \bar{\varphi}_N(\mathbf{x}) Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) D_k^T \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_{\Gamma_T} \int_{\Gamma_Q} D_i^T \bar{\varphi}_T(\mathbf{x}) Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) D_k^T \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} ds \\
& - \int_S \int_{\Gamma_Q} p(\mathbf{x}) [M_T^*(\mathbf{x}, \tilde{\mathbf{x}}) + R^*(\mathbf{x}, \tilde{\mathbf{x}})] \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} dS \\
& + \int_S \int_{\Gamma_Q} p(\mathbf{x}) Q^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{w}(\tilde{\mathbf{x}}) d\tilde{s} dS \\
& - \sum_d \overline{\Delta_d M_T} \int_{\Gamma_Q} [M_T^*(\mathbf{x}_c, \tilde{\mathbf{x}}) + R^*(\mathbf{x}_d, \tilde{\mathbf{x}})] \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} \quad (55)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_\Delta(\Delta \tilde{M}_T) = & - \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_{\Gamma_Q} \bar{Q}_K(\mathbf{x}) W^*(\mathbf{x}, \mathbf{x}_c) ds \\
& + \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_{\Gamma_M} \bar{M}_N(\mathbf{x}) \Phi_N^*(\mathbf{x}_c, \mathbf{x}) ds \\
& - \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_{\Gamma_N} \bar{\varphi}_N(\mathbf{x}) M_N^*(\mathbf{x}_c, \mathbf{x}) ds \\
& - \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_{\Gamma_T} \bar{\varphi}_T(\mathbf{x}) [M_T^*(\mathbf{x}_c, \mathbf{x}) + R^*(\mathbf{x}_c, \mathbf{x})] ds \\
& + \sum_{\tilde{c}} \tilde{w}(\mathbf{x}_c) \Delta_c \tilde{M}_T (1 - \kappa(\mathbf{x}_c)) \\
& - \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_S p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) dS \\
& - \sum_d \sum_{\tilde{c}} \overline{\Delta_d M_T} W^*(\mathbf{x}, \tilde{\mathbf{x}}) \Delta_{\tilde{c}} \tilde{M}_T \quad (56)
\end{aligned}$$

In establishing the above formulas, use has been made of the relations

$$P_{ij} n_i n_j = M_N, \quad P_{ij} \tau_i n_j = M_T + R$$

and, for convenience, of the operators

$$\begin{aligned}
D_a^N f(\mathbf{x}) & \stackrel{\text{Def}}{=} \frac{d}{ds} [n_a f](\mathbf{x}) = \left[n_a \frac{d}{ds} f + \frac{1}{\rho} \tau_a f \right](\mathbf{x}) \\
D_a^T f(\mathbf{x}) & \stackrel{\text{Def}}{=} \frac{d}{ds} [\tau_a f](\mathbf{x}) = \left[\tau_a \frac{d}{ds} f - \frac{1}{\rho} n_a f \right](\mathbf{x}) \quad (57)
\end{aligned}$$

The indices (c, \tilde{c}) (resp. d) range over those corners at which ΔM_T is unknown (resp. prescribed).

The symmetry properties of the various kernel functions imply that the variational formulation (46) is symmetric, i.e.

$$\mathcal{B}_{XY}(f, g) = \mathcal{B}_{YX}(g, f) \quad X, Y \in \{Q, M, N, T, \Delta\}$$

Besides, the domain integrals can be transformed into boundary integrals whenever the loading function $p(\mathbf{x})$ is harmonic, as is explained in Appendix B.

6 PARTICULAR BOUNDARY CONDITION CONFIGURATIONS

The formulation presented in the previous section holds for the most general types of boundary conditions. Let us now consider two specific cases: clamped plates and simply-supported plates.

6.1 Clamped plate

In this case, one has $\Gamma_M = \Gamma_Q = \emptyset$ and $\Gamma_N = \Gamma_T = \Gamma$ with $\bar{\varphi}_N = \bar{\varphi}_T = 0$; all twisting moment jumps are unknown. As a result, the formulation (46) reduces to

$$(\forall \tilde{Q}_K, \tilde{M}_N, \tilde{M}_T) \quad \begin{Bmatrix} \tilde{Q}_K \\ \tilde{M}_N \\ \Delta \tilde{M}_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{B}_{QQ} & \mathcal{B}_{QM} & \mathcal{B}_{Q\Delta} \\ \mathcal{B}_{MQ} & \mathcal{B}_{MM} & \mathcal{B}_{M\Delta} \\ \mathcal{B}_{\Delta Q} & \mathcal{B}_{\Delta M} & \mathcal{B}_{\Delta\Delta} \end{bmatrix} \begin{Bmatrix} Q_K \\ M_N \\ \Delta M_T \end{Bmatrix} = \begin{Bmatrix} \tilde{Q}_K \\ \tilde{M}_N \\ \Delta \tilde{M}_T \end{Bmatrix}^T \begin{Bmatrix} \mathcal{L}_Q \\ \mathcal{L}_M \\ \mathcal{L}_\Delta \end{Bmatrix} \quad (58)$$

with

$$\begin{aligned} \mathcal{L}_Q(\tilde{Q}_K) &= - \int_S \int_\Gamma p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} dS \\ \mathcal{L}_M(\tilde{M}_N) &= \int_S \int_\Gamma p(\mathbf{x}) W_{,i}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_i(\tilde{\mathbf{x}}) \tilde{M}_N(\tilde{\mathbf{x}}) d\tilde{s} dS \\ \mathcal{L}_\Delta(\Delta \tilde{M}_T) &= - \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_S p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) dS \end{aligned}$$

6.2 Simply-supported plate

In this case, one has $\Gamma_N = \Gamma_Q = \emptyset$ and $\Gamma_M = \Gamma_T = \Gamma$, with $\bar{M}_N = \bar{\varphi}_T = 0$; all twisting moment jumps are unknown. As a result, the formulation (46) reduces to

$$(\forall \tilde{Q}_K, \tilde{\varphi}_N, \tilde{M}_T) \quad \begin{Bmatrix} \tilde{Q}_K \\ \tilde{\varphi}_N \\ \Delta \tilde{M}_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{B}_{QQ} & \mathcal{B}_{QN} & \mathcal{B}_{Q\Delta} \\ \mathcal{B}_{NQ} & \mathcal{B}_{NN} & \mathcal{B}_{N\Delta} \\ \mathcal{B}_{\Delta Q} & \mathcal{B}_{\Delta N} & \mathcal{B}_{\Delta\Delta} \end{bmatrix} \begin{Bmatrix} Q_K \\ \varphi_N \\ \Delta M_T \end{Bmatrix} = \begin{Bmatrix} \tilde{Q}_K \\ \tilde{\varphi}_N \\ \Delta \tilde{M}_T \end{Bmatrix}^T \begin{Bmatrix} \mathcal{L}_Q \\ \mathcal{L}_N \\ \mathcal{L}_\Delta \end{Bmatrix} \quad (59)$$

with

$$\begin{aligned}\mathcal{L}_Q(\tilde{Q}_K) &= - \int_S \int_\Gamma p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{Q}_K(\tilde{\mathbf{x}}) d\tilde{s} dS \\ \mathcal{L}_N(\tilde{\varphi}_N) &= - \int_S \int_\Gamma p(\mathbf{x}) M_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} dS \\ \mathcal{L}_\Delta(\Delta \tilde{M}_T) &= - \sum_{\tilde{c}} \Delta_{\tilde{c}} \tilde{M}_T \int_S p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}_c) dS\end{aligned}$$

6.3 Free plate

In this case, one has $\Gamma_N = \Gamma_T = \emptyset$ and $\Gamma_M = \Gamma_Q = \Gamma$, with $\bar{M}_N = \bar{Q}_K = 0$; a zero value is prescribed for all twisting moment jumps. As a result, the formulation (46) reduces to

$$(\forall \tilde{\varphi}_N, \tilde{\varphi}_T) \quad \begin{Bmatrix} \tilde{\varphi}_N \\ \tilde{\varphi}_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{B}_{NN} & \mathcal{B}_{NT} \\ \mathcal{B}_{TN} & \mathcal{B}_{TT} \end{bmatrix} \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix} = \begin{Bmatrix} \tilde{\varphi}_N \\ \tilde{\varphi}_T \end{Bmatrix}^T \begin{Bmatrix} \mathcal{L}_N \\ \mathcal{L}_T \end{Bmatrix} \quad (60)$$

with

$$\begin{aligned}\mathcal{L}_N(\tilde{\varphi}_N) &= - \int_S \int_\Gamma p(\mathbf{x}) M_N^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{\varphi}_N(\tilde{\mathbf{x}}) d\tilde{s} dS \\ \mathcal{L}_T(\tilde{\varphi}_N) &= \int_S \int_\Gamma p(\mathbf{x}) [M_T^*(\mathbf{x}, \tilde{\mathbf{x}}) - R^*(\mathbf{x}, \tilde{\mathbf{x}})] \tilde{\varphi}_T(\tilde{\mathbf{x}}) d\tilde{s} dS\end{aligned}$$

The same formulation has been obtained by Giroire and Nedelec¹⁷, where, however, the possibility of corner points is not addressed.

7 NUMERICAL IMPLEMENTATION

A computer code has been implemented allowing the discretization of the plate boundary with either straight or circular arc elements (constant curvature elements). The vertical displacement field is modelled using hermitian cubic shape functions while the normal slope and the normal moment are approximated with lagrangian quadratic shape functions; both quadratic and linear shape functions have been tried for the Kirchhoff shear. The required C^1 (C^0 respectively) continuity of w (resp. φ_N, M_N, Q_K) at a smooth point on the boundary is thus easily enforced.

At a corner node, the enforcement of the cartesian gradient continuity across elements is achieved through the supplementary conditions

$$n_i^1 \Phi_N^1 + \tau_i^1 \Phi_T^1 = n_i^2 \Phi_N^2 + \tau_i^2 \Phi_T^2$$

where indices 1 and 2 pertain to the relevant element and Φ_T and Φ_N are the nodal values of the tangential and normal slopes. Besides, the static boundary variables M_N, Q_K are expected to jump across either corners or endpoints of Γ_M or Γ_Q .

The system of equations (46) requires at most the evaluation of double logarithmic-singular integrals. Any such integral may be reduced to the form

$$I = \int_{-1}^1 \int_{-1}^1 z(\xi, \eta) \ln[r(\xi, \eta)] d\xi d\eta$$

where $z(\xi, \eta)$ is a regular function and ξ and η are intrinsic coordinates. Basically, three different cases have to be dealt with: integration over separate elements (no singularity), integration over adjacent elements (singularity at $\xi = 1, \eta = -1$ or $\xi = -1, \eta = 1$ depending on the relative position of the two elements) and integration over coincident elements (singularity along the diagonal $\xi = \eta$). In the second case, as shown by Parreira and Guiggiani³⁶ or Frangi and Novati¹⁵, simple Gauss-Legendre quadrature rules can be used after introducing suitable coordinate transformations concentrating sample points near the singularity. In the last case the coordinate transformations

$$\begin{cases} \eta = \alpha - \beta - \alpha\beta \\ \xi = \alpha + \beta - \alpha\beta \end{cases} \quad (\eta \geq \xi,) \qquad \begin{cases} \eta = \alpha + \beta - \alpha\beta \\ \xi = \alpha - \beta - \alpha\beta \end{cases} \quad (\eta \leq \xi)$$

lead to

$$I = \int_{-1}^1 \int_{-1}^1 z(\xi, \eta) \ln \frac{2r(\xi, \eta)}{|\xi - \eta|} d\xi d\eta + 2 \int_0^1 (1 - \beta) \ln \beta \int_{-1}^1 [z(\alpha, \beta) + z(\alpha, -\beta)] d\alpha d\beta$$

The first integral is regular, while the second one can be evaluated by means of Gauss-Legendre (for the inner integral) and Gauss logarithmic (for the outer integral) quadrature rules.

8 NUMERICAL EXAMPLES

In this section, some numerical examples are presented for simple situations where an analytical solution is available for comparison. Since one of the authors also published recently a paper¹⁴ on the regularized collocation approach to BEM formulation (CBEM) for Kirchhoff plate bending, some comparisons to the CBEM numerical results are presented for completeness.

Examples 1–3: square plate. A square plate (side length $a = 1$, $\nu = .3$, $D = 1$) is subjected to a uniform pressure $p = 1$. Four different meshes (labelled A, B, C and D), having respectively two, four, eight and twenty elements on each side, are considered. Thanks to geometrical and loading symmetry, results will be shown for a half-side. Three different sets of boundary conditions are considered:

1. *Clamped plate.* Following Stern⁴² and Williams⁴⁷, the normal moment at corners is imposed equal to zero in this case.

2. *Figure 1 should appear here.*

Figure 2 should appear here.

Table 1 should appear here.

Table 2 should appear here.

In Figures (1) and (2), the numerical results for the normal moment and the shear are plotted against the analytical solution of Timoshenko and Woinowsky-Krieger⁴⁴. Table 1 displays numerical values of L^2 relative average errors between computed and analytical nodal values of the relevant boundary variables, while table 2 shows the numerical values found for the concentrated forces (jumps $\Delta_c M_T$ of twisting moment) at corners, which vanish in the exact solution.

3. *Simply-supported plate.* An analysis of the asymptotic behaviour of the solution at corner points (Williams⁴⁷), or the imposition of additional relations at corners (Guoshu and Mukherjee²⁰), shows that the equivalent shear Q_K at corners must vanish; this additional condition has been used for the present computations.

Figure 3 should appear here.

Figure 4 should appear here.

Table 3 should appear here.

Table 4 should appear here.

In Figures 3 and 3, the numerical results for the normal slope and the shear are plotted against the analytical solution⁴⁴. Table 3 displays numerical values of L^2 relative average errors between computed and analytical nodal values of the relevant boundary variables, while table 4 shows the numerical values found for the $\Delta_c M_T$.

4. *Plate with two opposite sides simply-supported and the other ones free.* In this case, for which an analytical solution is also known⁴⁴, no additional hypothesis needs to be introduced for the unknowns at corner points. From the boundary element formulation viewpoint, this set of boundary conditions is the most delicate one since the kernels of highest singularities are involved in the original (i.e. non-regularized) formulation.

Figure 5 should appear here.

Figure 6 should appear here.

Figure 7 should appear here.

Table 6 should appear here.

Table 5 should appear here.

Figure 5 displays the numerical results for the normal slope, while table 5 shows the L^2 relative average errors between computed and analytical nodal values of the relevant unknowns. Concerning the equivalent shear Q_K , the results using quadratic interpolation (labelled ‘3-noded’) show a strong tendency to spatial oscillations, see figure 6, and are poor for the coarser meshes A and B; also, poor results and slow convergence with mesh refinement is observed for the corner forces, see table 6. The same oscillatory tendency, with somewhat milder impact, is observed on the numerical results obtained by collocation BEM¹⁴. However, when a piecewise linear interpolation (labelled ‘2-noded’) is used instead for Q_K , the results for both shear and corner forces improve dramatically (see figure 7 and table 6).

Example 4: Simply-supported circular plate. A circular (with radius $R = 1$) simply-supported plate subjected to a uniform pressure $p = 1$ is considered ($\nu = .3$, $D = 1$). Circular-arc shaped elements are used, allowing an exact representation of the problem geometry. The analytical solution is known:

$$Q_K(\rho) = -p_0 \frac{R}{2} \rho, \quad \varphi_N = \frac{p_0 R^3}{16D} \rho \left(\rho^2 - \frac{3 + \nu}{1 + \nu} \right)$$

where $\rho = r/R$ and r is the distance from the center of the plate. The comparison is shown in table 7 for the boundary variables, which are constant due to axisymmetry. Using eight elements of equal size, an excellent agreement with the exact solution is reached.

Table 7 should appear here.

Galerkin vs. collocation. As far as relative accuracy is considered, both Galerkin and collocation approaches give good results, and neither outperforms the other one in all cases, as is apparent in tables 1–6.

As of yet, neither method has been optimized with respect to the respective numerical integration procedures; in particular the number of Gauss points is not currently adjusted to the relative interelement distance. For this reason, it is not easy to formulate meaningful conclusions concerning the respective computational efficiencies. A typical value of the integration time ratio observed between Galerkin and collocation BEM formulations is 1.5. On the other hand, the optimization of the Gauss point number for both methods is expected should logically reduce the integration computer time by a bigger amount for the SGBEM, where double integrals are computed instead of the single integrals of the CBEM. Also, the final linear system of equations obtained by the SGBEM is symmetric. Thus both the computer solution time and the storage required are half those entailed by the CBEM. This means that a SGBEM is, in an asymptotic sense (i.e. for sufficiently fine meshes), computationally more efficient than a CBEM; this has been pointed out for elasticity³.

9 STIFFNESS MATRIX OF AN ELASTIC KIRCHHOFF PLATE

In the context of coupled BEM /FEM approaches, the stiffness matrix of the elastic, BEM-modelled, transverse load-free (i.e. $p = 0$) part of the plate can be computed using the SGBE approach. Indeed, let $(\varphi_N, \varphi_T, M_N, M_T, \Delta M_T)$ denote any compatible set of boundary variables (i.e. the boundary variables associated to a solution w to (4)).

First, considering the moment M_N , shear Q_K and jumps ΔM_T as induced by prescribed values of φ_N, φ_T (i.e. $\Gamma_N = \Gamma_T = \Gamma$, $\Gamma_M = \Gamma_Q = \emptyset$), the formulation (46) takes the form:

$$(\forall \tilde{Q}_K, \tilde{M}_N, \tilde{M}_T) \left\{ \begin{array}{c} \tilde{Q}_K \\ \tilde{M}_N \\ \Delta \tilde{M}_T \end{array} \right\}^T \left[\begin{array}{ccc} \mathcal{B}_{QQ} & \mathcal{B}_{QM} & \mathcal{B}_{Q\Delta} \\ \mathcal{B}_{MQ} & \mathcal{B}_{MM} & \mathcal{B}_{M\Delta} \\ \mathcal{B}_{\Delta Q} & \mathcal{B}_{\Delta M} & \mathcal{B}_{\Delta\Delta} \end{array} \right] \left\{ \begin{array}{c} Q_K \\ M_N \\ \Delta M_T \end{array} \right\} = \left\{ \begin{array}{c} \tilde{Q}_K \\ \tilde{M}_N \\ \Delta \tilde{M}_T \end{array} \right\}^T \left[\begin{array}{cc} \mathcal{L}_{QN} & \mathcal{L}_{QT} \\ \mathcal{L}_{MN} & \mathcal{L}_{MT} \\ \mathcal{L}_{\Delta N} & \mathcal{L}_{\Delta T} \end{array} \right] \left\{ \begin{array}{c} \varphi_N \\ \varphi_T \end{array} \right\} \quad (61)$$

where the linearity of the right-hand side with respect to φ_N, φ_T is emphasized.

Then, considering the normal and tangential gradients φ_N, φ_T as induced by prescribed moment M_N , shear Q_K and twisting moment jumps ΔM_T (i.e. $\Gamma_M = \Gamma_Q = \Gamma$, $\Gamma_N = \Gamma_T = \emptyset$), the formulation (46) becomes:

$$(\forall \tilde{\varphi}_N, \tilde{\varphi}_T, \tilde{M}_T)$$

$$\begin{Bmatrix} \tilde{\varphi}_N \\ \tilde{\varphi}_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{B}_{NN} & \mathcal{B}_{NT} \\ \mathcal{B}_{TN} & \mathcal{B}_{TT} \end{bmatrix} \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix} = \begin{Bmatrix} \tilde{\varphi}_N \\ \tilde{\varphi}_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{L}_{NQ} & \mathcal{L}_{NM} & \mathcal{L}_{N\Delta} \\ \mathcal{L}_{TQ} & \mathcal{L}_{TM} & \mathcal{L}_{T\Delta} \end{bmatrix} \begin{Bmatrix} Q_K \\ M_N \\ \Delta M_T \end{Bmatrix} \quad (62)$$

Now, the test functions are particularized: let $\tilde{M}_N = M_N$, $\tilde{Q}_K = Q_K$, $\Delta \tilde{M}_T = \Delta M_T$ in (61) and $\tilde{\varphi}_N = \varphi_N$, $\tilde{\varphi}_T = \varphi_T$ in (61). Upon detailed inspection of eqs. (52) to (56), one can show that:

$$\begin{aligned} & \begin{Bmatrix} Q_K \\ M_N \\ \Delta M_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{L}_{QN} & \mathcal{L}_{QT} \\ \mathcal{L}_{MN} & \mathcal{L}_{MT} \\ \mathcal{L}_{\Delta N} & \mathcal{L}_{\Delta T} \end{bmatrix} \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix} - \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{L}_{NQ} & \mathcal{L}_{NM} & \mathcal{L}_{N\Delta} \\ \mathcal{L}_{TQ} & \mathcal{L}_{TM} & \mathcal{L}_{T\Delta} \end{bmatrix} \begin{Bmatrix} Q_K \\ M_N \\ \Delta M_T \end{Bmatrix} \\ &= \int_{\Gamma} [w(\mathbf{x})Q_K(\mathbf{x}) - \varphi_T(\mathbf{x})M_T(\mathbf{x})] ds + \sum_c w(\mathbf{x}_c)\Delta_c M_T \quad (63) \end{aligned}$$

where the summation ranges over all corners. The right-hand side of the above equation is twice the strain energy W associated with a bending displacement field w over S which solves the homogeneous elastic equilibrium equation (9):

$$W = \frac{1}{2} \int_S w_{,ij} K_{ijkl} w_{,kl} dS = \frac{1}{2} \int_{\Gamma} [w(\mathbf{x})Q_K(\mathbf{x}) - \varphi_T(\mathbf{x})M_T(\mathbf{x})] ds + \frac{1}{2} \sum_c w(\mathbf{x}_c)\Delta_c M_T$$

That fact can easily be established from equation (8) and using (6) and (12). In view of the equalities (61,62), the above result means that the strain energy is also given by

$$2W = \begin{Bmatrix} Q_K \\ M_N \\ \Delta M_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{B}_{QQ} & \mathcal{B}_{QM} & \mathcal{B}_{Q\Delta} \\ \mathcal{B}_{MQ} & \mathcal{B}_{MM} & \mathcal{B}_{M\Delta} \\ \mathcal{B}_{\Delta Q} & \mathcal{B}_{\Delta M} & \mathcal{B}_{\Delta\Delta} \end{bmatrix} \begin{Bmatrix} Q_K \\ M_N \\ \Delta M_T \end{Bmatrix} - \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{B}_{NN} & \mathcal{B}_{NT} \\ \mathcal{B}_{TN} & \mathcal{B}_{TT} \end{bmatrix} \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix} \quad (64)$$

$$= 2 \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix}^T \begin{bmatrix} \mathcal{K}_{NN} & \mathcal{K}_{NT} \\ \mathcal{K}_{TN} & \mathcal{K}_{TT} \end{bmatrix} \begin{Bmatrix} \varphi_N \\ \varphi_T \end{Bmatrix} \quad (65)$$

where $Q_K, M_N, \Delta M_T$ have been expressed in terms of φ_N, φ_T using equation (61), so that the stiffness operator \mathcal{K} is given by:

$$2 \begin{bmatrix} \mathcal{K}_{NN} & \mathcal{K}_{NT} \\ \mathcal{K}_{TN} & \mathcal{K}_{TT} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{QN} & \mathcal{L}_{QT} \\ \mathcal{L}_{MN} & \mathcal{L}_{MT} \\ \mathcal{L}_{\Delta N} & \mathcal{L}_{\Delta T} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_{QQ} & \mathcal{B}_{QM} & \mathcal{B}_{Q\Delta} \\ \mathcal{B}_{MQ} & \mathcal{B}_{MM} & \mathcal{B}_{M\Delta} \\ \mathcal{B}_{\Delta Q} & \mathcal{B}_{\Delta M} & \mathcal{B}_{\Delta\Delta} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{L}_{QN} & \mathcal{L}_{QT} \\ \mathcal{L}_{MN} & \mathcal{L}_{MT} \\ \mathcal{L}_{\Delta N} & \mathcal{L}_{\Delta T} \end{bmatrix} - \begin{bmatrix} \mathcal{B}_{NN} & \mathcal{B}_{NT} \\ \mathcal{B}_{TN} & \mathcal{B}_{TT} \end{bmatrix} \quad (66)$$

The above result is interesting in that the stiffness matrix of an elastic plate without transverse load (i.e. $p = 0$) is expressed in terms of the boundary displacement and slope variables, i.e. the trace on the boundary of the variables that appear naturally in a FEM modelling. Hence, a coupled BEM / FEM approach for complex systems is available, whereby (for instance) elastic and unloaded subregions are treated as ‘macro-elements’ in a variational formulation using equation (66) for the relevant stiffness matrices.

10 COMMENTS

In this closing section, after briefly commenting on the various approaches leading to SGBE formulations, some applications and extensions of the present approach are briefly discussed.

Variational approach vs. weighted residuals. The variational viewpoint has been adopted in this paper, i.e. the Galerkin formulation is formulated here as the stationarity condition of the potential energy applied to adequately chosen test functions. Besides, the derivation of test functions δw using the integral representation approach would become very awkward were they to verify the homogeneous kinematic boundary conditions in addition to the local equilibrium. This led to introduce the kinematic boundary conditions explicitly, augmenting the potential energy functional with constraint terms.

However, the test function δw , eq. (23) can be formulated directly, interpreting $\tilde{w}, \tilde{M}_N, \tilde{Q}_K, \Delta \tilde{M}_T$ as kinematic and static source distributions. The same final Galerkin direct BIE formulation is then achieved by directly substituting δw , etc. into the reciprocity identity (13); this is essentially the approach followed in some earlier papers on elastic problems, e.g. Sirtori *et al.*⁴⁰. In turn, the ‘source distribution’ approach is yet just another way to formulate the boundary integral equation in weighted residual form.

Summing up, the weighted residual, source distribution and variational viewpoints leads to the same Galerkin BIE formulation. The main justification for our adopting the latter lies in the extra insight gained about the basic principles underlying the formulation.

Other applications. The SGBEM approach has specific advantages in some specific situations. In particular, as shown in Sec. 9 above, it plays a crucial role in formulating the stiffness matrix of a given region in terms of boundary kinematical variables.

Besides, SGBIE formulations are useful in energy methods for fracture mechanics. Specifically, the energy release rate G for a cracked elastic body can be formulated as (minus) the kernel of the domain derivative of the potential energy at equilibrium with respect to (virtual) crack extensions. It has been shown¹⁰, for three-dimensional elastic bodies, that a combined use of SGBIE and material differentiation allows to compute G without having to compute the first-order domain derivative of the equilibrium elastic solution, the latter being eliminated by virtue of the symmetry of the governing elastic formulation. This idea has been recently pursued further for cracked Kirchhoff plates¹⁶, using the present SGBIE formulation; satisfactory numerical results have been obtained.

The present SGBIE approach is also adaptable to elastic plates resting on Winkler foundations, i.e. with a transverse load $p - kw$, where k is the foundation stiffness. The initial stationarity statement (13) must be modified accordingly. A fundamental solution for the differential operator $Dw_{,ijj} + kw$ (k constant) is given³⁸ by

$$W_{winkler}^*(\mathbf{x}, \tilde{\mathbf{x}}) = -\frac{1}{2\pi(kD)^{1/2}} \text{kei}\left(\sqrt[4]{\frac{k}{D}}r\right) = W^*(\mathbf{x}, \tilde{\mathbf{x}}) + (\text{higher order terms})$$

where kei denotes the Kelvin function of order zero. The difference between the fundamental solutions $W_{winkler}^*(\mathbf{x}, \tilde{\mathbf{x}})$ and $W^*(\mathbf{x}, \tilde{\mathbf{x}})$ or their derivatives of any order is nonsingular. Hence, the integrations by parts techniques used here can be applied to the SGBIE formulation of Winkler plates as well, using the splitting $W_{winkler}^* = W^* + (W_{winkler}^* - W^*)$.

A Galerkin formulation of the domain-BIE approach to Winkler plates⁸, using the usual fundamental solution W^* , eq. (17), is less straightforward due to the fact that the test functions (23) do not solve the homogeneous partial differential equation for Winkler plates. Thus, an additional, non-trivial, investigation is necessary for the symmetrization of the domain-BIE approach. The multiple reciprocity approach proposed by Sladek and Sladek⁴¹ should prove helpful for this development.

11 CONCLUSIONS

A symmetric BE method for linear elastic Kirchhoff plates has been developed. The formulation stems from the imposition of the stationarity conditions of an augmented potential energy functional to those test functions that satisfy the homogeneous local elastic equilibrium equation. A regularization approach based on integrations by parts has been developed, so that the governing bilinear form in the final variational integral formulation involves a weakly singular boundary integral followed by a nonsingular boundary integral. The formulation is valid for an arbitrary plate shape and arbitrary boundary conditions. A BE implementation has been developed. Numerical results, on examples involving rectangular or circular plates under uniform pressure, exhibit very good accuracy when compared to exact solutions. The stiffness matrix of an elastic plate with $p = 0$ has been expressed in terms of the kinematical variables on the plate boundary, for e.g. coupling or subregion purposes.

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A DETERMINATION OF THE REGULARIZED KERNELS

A.1 The R^* kernel

Let us consider a (r, θ) polar coordinate system centered at \mathbf{x} , so that $\mathbf{r} = \tilde{\mathbf{x}} - \mathbf{x} = r\mathbf{e}_r$ and $\tilde{\nabla}r = \mathbf{e}_r$. Introducing a parametrization $(r(\tilde{s}), \theta(\tilde{s}))$ of the closed curve $\tilde{\Gamma}$ in terms of

the arc-length \tilde{s} , the unit normal on $\tilde{\Gamma}$ is then given by

$$\tilde{\mathbf{n}} = -\frac{dr}{d\tilde{s}}\mathbf{e}_\theta + \frac{rd\theta}{d\tilde{s}}\mathbf{e}_r$$

Thus, from definition (20):

$$Q^*(\mathbf{x}, \tilde{\mathbf{x}}) = -\frac{1}{2\pi} \frac{\tilde{\mathbf{n}} \cdot \mathbf{e}_r}{r} = -\frac{1}{2\pi} \frac{d\theta}{d\tilde{s}} = \frac{d}{d\tilde{s}} R^*(\mathbf{x}, \tilde{\mathbf{x}}) \quad (67)$$

which establishes the expression equation (26) of $R^*(\mathbf{x}, \tilde{\mathbf{x}})$. Moreover, since

$$\tilde{\nabla}\theta = \frac{1}{r}\mathbf{e}_\theta = \frac{1}{r}(\mathbf{e}_z \wedge \mathbf{e}_r)$$

it follows that

$$R_{,\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) = e_{ij} DW_{,aa\tilde{j}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \quad R_{,\tilde{ii}}^*(\mathbf{x}, \tilde{\mathbf{x}}) = 0 \quad (68)$$

Finally, one notes that

$$R^*(\tilde{\mathbf{x}}, \mathbf{x}) = R^*(\mathbf{x}, \tilde{\mathbf{x}}) + \pi, \quad R_{,\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) = -R_{,i}^*(\mathbf{x}, \tilde{\mathbf{x}}) \quad (69)$$

A.2 Integration of $P_{ij,\tilde{k}}^* \tilde{n}_j$

First, using (68), one has

$$M_{ij,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) - R_{,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) e_{ij} = M_{ij,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) - e_{ij} e_{k\ell} DW_{,aa\tilde{\ell}}^*(\mathbf{x}, \tilde{\mathbf{x}})$$

so that, using the identity $e_{ij} e_{kp} = e_{ij3} e_{kp3} = \delta_{ik} \delta_{jp} - \delta_{ip} \delta_{jk}$

$$\begin{aligned} \left[M_{ij,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) - R_{,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) e_{ij} \right] \tilde{n}_j &= M_{ij,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j + DW_{,aa\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_k - \delta_{ik} DW_{,aa\tilde{j}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j \\ &= M_{ij,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j - M_{ij,\tilde{j}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_k + \delta_{ik} Q^*(\mathbf{x}, \tilde{\mathbf{x}}) \end{aligned} \quad (70)$$

Then, using (67), $\delta_{ik} = e_{ij} e_{kj}$, and the identity

$$\tilde{n}_j f_{,\tilde{k}} - \tilde{n}_k f_{,\tilde{j}} = e_{jk} \frac{df}{d\tilde{s}} \quad (71)$$

equation (70) readily leads to

$$P_{ij,\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \tilde{n}_j(\tilde{\mathbf{x}}) = -e_{kj} \frac{d}{d\tilde{s}} P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}})$$

and hence to the desired result, equation (33).

A.3 Integration of $K_{ijab} P_{kl,ab}^* n_i$

First, it is easy to show that the following symmetry property holds:

$$K_{ijab} M_{kl,ab}^*(\mathbf{x}, \tilde{\mathbf{x}}) = K_{klab} M_{ij,ab}^*(\mathbf{x}, \tilde{\mathbf{x}})$$

so that one has

$$\begin{aligned}
& K_{ijab} P_{kl,abj}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_i(\mathbf{x}) \\
&= K_{klab} M_{ij,abj}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_i(\mathbf{x}) - K_{ijab} e_{kl} R_{,abj}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_i(\mathbf{x}) \\
&= K_{klab} (M_{ij,abj}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_i(\mathbf{x}) - M_{ij,ijb}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_a(\mathbf{x})) - D e_{kl} R_{,aai}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_i(\mathbf{x}) \\
&= e_{ai} K_{klab} \frac{d}{ds} M_{ij,bj}^* (\mathbf{x}, \tilde{\mathbf{x}}) \\
&= K_{klab} \frac{d}{ds} R_{,ab}^* (\mathbf{x}, \tilde{\mathbf{x}})
\end{aligned}$$

using (68) and (71). Thus the desired result (41) is established.

A.4 The Z_{ik} kernel

The aim of this section is to find a kernel $Z_{ik}^* (\mathbf{x}, \tilde{\mathbf{x}})$ such that

$$A_{ik}^* (\mathbf{x}, \tilde{\mathbf{x}}) \equiv [e_{ij} K_{klab} R_{,ab}^* (\mathbf{x}, \tilde{\mathbf{x}}) - K_{ijab} P_{kl,ab}^* (\mathbf{x}, \tilde{\mathbf{x}})] n_j(\mathbf{x}) \tilde{n}_\ell(\tilde{\mathbf{x}}) = \frac{d}{ds} \frac{d}{d\tilde{s}} Z_{ik}^* (\mathbf{x}, \tilde{\mathbf{x}})$$

First, the definition of $P_{kl}^* (\mathbf{x}, \tilde{\mathbf{x}})$ implies that

$$A_{ik}^* (\mathbf{x}, \tilde{\mathbf{x}}) = [e_{ij} K_{klab} R_{,ab}^* (\mathbf{x}, \tilde{\mathbf{x}}) + e_{kl} K_{ijab} R_{,ab}^* (\mathbf{x}, \tilde{\mathbf{x}}) - K_{ijab} M_{kl,ab}^* (\mathbf{x}, \tilde{\mathbf{x}})] n_j(\mathbf{x}) \tilde{n}_\ell(\tilde{\mathbf{x}})$$

Then, using the elastic constitutive law, equations (2)–(3), one obtains

$$-K_{ijab} M_{kl,ab}^* (\mathbf{x}, \tilde{\mathbf{x}}) = D^2 (1 - \nu) \left[(1 - \nu) W_{,ij\tilde{k}\tilde{\ell}}^* (\mathbf{x}, \tilde{\mathbf{x}}) + \nu \delta_{ij} W_{,aa\tilde{k}\tilde{\ell}}^* (\mathbf{x}, \tilde{\mathbf{x}}) + \nu \delta_{kl} W_{,ij\tilde{a}\tilde{a}}^* (\mathbf{x}, \tilde{\mathbf{x}}) \right]$$

where the equilibrium equation $W_{,a\tilde{a}\tilde{b}\tilde{b}}^* (\mathbf{x}, \tilde{\mathbf{x}}) = W_{,\tilde{a}\tilde{a}\tilde{b}\tilde{b}}^* (\mathbf{x}, \tilde{\mathbf{x}}) = 0$ has been used. Now, repeated applications of (71) give, again using $W_{,a\tilde{a}\tilde{b}\tilde{b}}^* (\mathbf{x}, \tilde{\mathbf{x}}) = 0$

$$-K_{ijab} M_{kl,ab}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_j(\mathbf{x}) \tilde{n}_\ell(\tilde{\mathbf{x}})$$

ex

$$\begin{aligned}
&= D^2 (1 - \nu) \left[e_{ij} \tilde{n}_k(\tilde{\mathbf{x}}) \frac{d}{ds} W_{,j\tilde{\ell}\tilde{\ell}}^* (\mathbf{x}, \tilde{\mathbf{x}}) + e_{kl} n_i(\mathbf{x}) \frac{d}{d\tilde{s}} W_{,jj\tilde{\ell}}^* (\mathbf{x}, \tilde{\mathbf{x}}) \right. \\
&\quad \left. + (1 - \nu) e_{ij} e_{kl} \frac{d}{ds} \frac{d}{d\tilde{s}} W_{,j\tilde{\ell}}^* (\mathbf{x}, \tilde{\mathbf{x}}) \right] \quad (72)
\end{aligned}$$

while, using properties (68) and (69) of the kernel $R^* (\mathbf{x}, \tilde{\mathbf{x}})$ and its derivatives, one has

$$\begin{aligned}
& \frac{1}{D(1 - \nu)} e_{ij} K_{klab} R_{,ab}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_j(\mathbf{x}) \tilde{n}_\ell(\tilde{\mathbf{x}}) \\
&= e_{ij} R_{,\tilde{k}\tilde{\ell}}^* (\mathbf{x}, \tilde{\mathbf{x}}) n_j(\mathbf{x}) \tilde{n}_\ell(\tilde{\mathbf{x}}) \\
&= -e_{ij} e_{kl} n_j(\mathbf{x}) \frac{d}{d\tilde{s}} R_{,\tilde{\ell}}^* (\mathbf{x}, \tilde{\mathbf{x}}) \\
&= D e_{kl} n_j(\mathbf{x}) e_{ij} e_{lm} \frac{d}{d\tilde{s}} W_{,a\tilde{a}\tilde{m}}^* (\mathbf{x}, \tilde{\mathbf{x}}) \\
&= D e_{kl} \left[\delta_{i\ell} n_m(\mathbf{x}) \frac{d}{d\tilde{s}} W_{,a\tilde{a}\tilde{m}}^* (\mathbf{x}, \tilde{\mathbf{x}}) - \tilde{n}_\ell(\tilde{\mathbf{x}}) \frac{d}{d\tilde{s}} W_{,a\tilde{a}i}^* (\mathbf{x}, \tilde{\mathbf{x}}) \right] \\
&= e_{ki} \frac{d}{ds} \frac{d}{d\tilde{s}} R^* (\mathbf{x}, \tilde{\mathbf{x}}) - D e_{kl} \tilde{n}_\ell(\tilde{\mathbf{x}}) \frac{d}{d\tilde{s}} W_{,a\tilde{a}i}^* (\mathbf{x}, \tilde{\mathbf{x}}) \quad (73)
\end{aligned}$$

and, similarly

$$\begin{aligned} \frac{1}{D(1-\nu)} e_{k\ell} K_{ijab} R_{,ab}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_j(\mathbf{x}) \tilde{n}_\ell(\tilde{\mathbf{x}}) \\ = e_{ik} \frac{d}{ds} \frac{d}{d\tilde{s}} R^*(\mathbf{x}, \tilde{\mathbf{x}}) - D e_{ij} \tilde{n}_j(\tilde{\mathbf{x}}) \frac{d}{ds} W_{,a\tilde{a}\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \end{aligned} \quad (74)$$

Finally, substituting equations (72), (73), (74) and since $e_{ki} + e_{ik} = 0$, one finds

$$\begin{aligned} A_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) &= D^2(1-\nu) e_{ij} \left[\tilde{n}_k(\tilde{\mathbf{x}}) \frac{d}{ds} W_{,j\tilde{\ell}\tilde{\ell}}^*(\mathbf{x}, \tilde{\mathbf{x}}) - \tilde{n}_j(\tilde{\mathbf{x}}) \frac{d}{ds} W_{,a\tilde{a}\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \right] \\ &\quad + D^2(1-\nu) e_{k\ell} \left[n_i(\mathbf{x}) \frac{d}{d\tilde{s}} W_{,jj\tilde{\ell}}^*(\mathbf{x}, \tilde{\mathbf{x}}) - n_\ell(\mathbf{x}) \frac{d}{d\tilde{s}} W_{,a\tilde{a}i}^*(\mathbf{x}, \tilde{\mathbf{x}}) \right] \\ &\quad + D^2(1-\nu)^2 e_{ij} e_{k\ell} \frac{d}{ds} \frac{d}{d\tilde{s}} W_{,j\tilde{\ell}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \\ &= D^2 \left[(1-\nu)^2 e_{ij} e_{k\ell} \frac{d}{ds} \frac{d}{d\tilde{s}} W_{,j\tilde{\ell}}^*(\mathbf{x}, \tilde{\mathbf{x}}) - 2\delta_{ik}(1-\nu) \frac{d}{ds} \frac{d}{d\tilde{s}} W_{,a\tilde{a}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \right] \end{aligned} \quad (75)$$

which readily leads to the desired result

$$\begin{aligned} A_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) &= \frac{d}{ds} \frac{d}{d\tilde{s}} Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) \\ Z_{ik}^*(\mathbf{x}, \tilde{\mathbf{x}}) &= -D^2 \left[(1-\nu^2) \delta_{ik} W_{,a\tilde{a}}^*(\mathbf{x}, \tilde{\mathbf{x}}) + (1-\nu)^2 W_{,i\tilde{k}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \right] \end{aligned}$$

B CONVERSION OF DOMAIN INTEGRALS INTO BOUNDARY INTEGRALS

In the case of uniform pressure acting on the plate, the domain integrals may be transformed into contour integrals, following Balas and Sládek⁴.

It is easy to show⁴ that one has

$$W^*(\mathbf{x}, \tilde{\mathbf{x}}) = F_{,aa}^*(\mathbf{x}, \tilde{\mathbf{x}}) \quad \text{with} \quad F^*(\mathbf{x}, \tilde{\mathbf{x}}) = \frac{1}{128\pi D} r^4 (\ln r - 1)$$

On the other hand, the third Green's formula leads to

$$\begin{aligned} \int_S p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}) dS &= \int_S p_{,aa}(\mathbf{x}) F^*(\mathbf{x}, \tilde{\mathbf{x}}) dS \\ &\quad + \int_\Gamma \{ p(\mathbf{x}) F_{,a}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_a(\mathbf{x}) - p_{,a}(\mathbf{x}) n_a(\mathbf{x}) F^*(\mathbf{x}, \tilde{\mathbf{x}}) \} ds \end{aligned}$$

Thus, for any load $p(\mathbf{x})$ which is a harmonic function, the above identity converts $\int_S p W dS$ into boundary integrals. For the special case of a uniform load, $p(\mathbf{x}) = p_0$, one gets

$$\int_S p(\mathbf{x}) W^*(\mathbf{x}, \tilde{\mathbf{x}}) dS = p_0 \int_\Gamma F_{,a}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_a(\mathbf{x}) ds$$

In a similar manner, one has

$$\begin{aligned}
\int_S p(\mathbf{x}) W_{,\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \, dS &= \int_S p_{,aa}(\mathbf{x}) F_{,\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \, dS \\
&\quad + \int_{\Gamma} \{ p(\mathbf{x}) F_{,a\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_a(\mathbf{x}) - p_{,a}(\mathbf{x}) n_a(\mathbf{x}) F_{,\tilde{i}}^*(\mathbf{x}, \tilde{\mathbf{x}}) \} \, ds \\
\int_S p(\mathbf{x}) M_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \, dS &= -K_{ijkl} \int_S p_{,aa}(\mathbf{x}) F_{,kl}^*(\mathbf{x}, \tilde{\mathbf{x}}) \, dS \\
&\quad - K_{ijkl} \int_{\Gamma} \{ p(\mathbf{x}) F_{,akl}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_a(\mathbf{x}) - p_{,a}(\mathbf{x}) n_a(\mathbf{x}) F_{,kl}^*(\mathbf{x}, \tilde{\mathbf{x}}) \} \, ds
\end{aligned}$$

Finally, using (68) it is easy to show that

$$R^*(\mathbf{x}, \tilde{\mathbf{x}}) = G_{,aa}^*(\mathbf{x}, \tilde{\mathbf{x}}), \quad \text{with } G^*(\mathbf{x}, \tilde{\mathbf{x}}) = \frac{1}{4} r^2 R^*(\mathbf{x}, \tilde{\mathbf{x}})$$

As a consequence, the kernel

$$H_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) = -K_{ijkl} F_{,kl}^*(\mathbf{x}, \tilde{\mathbf{x}}) - e_{ij} G_{,aa}^*(\mathbf{x}, \tilde{\mathbf{x}})$$

is such that

$$H_{ij,aa}^*(\mathbf{x}, \tilde{\mathbf{x}}) = P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}})$$

and one has

$$\begin{aligned}
\int_S p(\mathbf{x}) P_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \, dS &= \int_S p_{,aa}(\mathbf{x}) H_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \, dS \\
&\quad + \int_{\Gamma} \{ p(\mathbf{x}) H_{ij,a}^*(\mathbf{x}, \tilde{\mathbf{x}}) n_a(\mathbf{x}) - p_{,a}(\mathbf{x}) n_a(\mathbf{x}) H_{ij}^*(\mathbf{x}, \tilde{\mathbf{x}}) \} \, ds
\end{aligned}$$

Finally, if the load $p(\mathbf{x})$ is not harmonic, a multiple reciprocity approach (Nowak and Brebbia³⁴) could be developed by using repeatedly the above line of reasoning.

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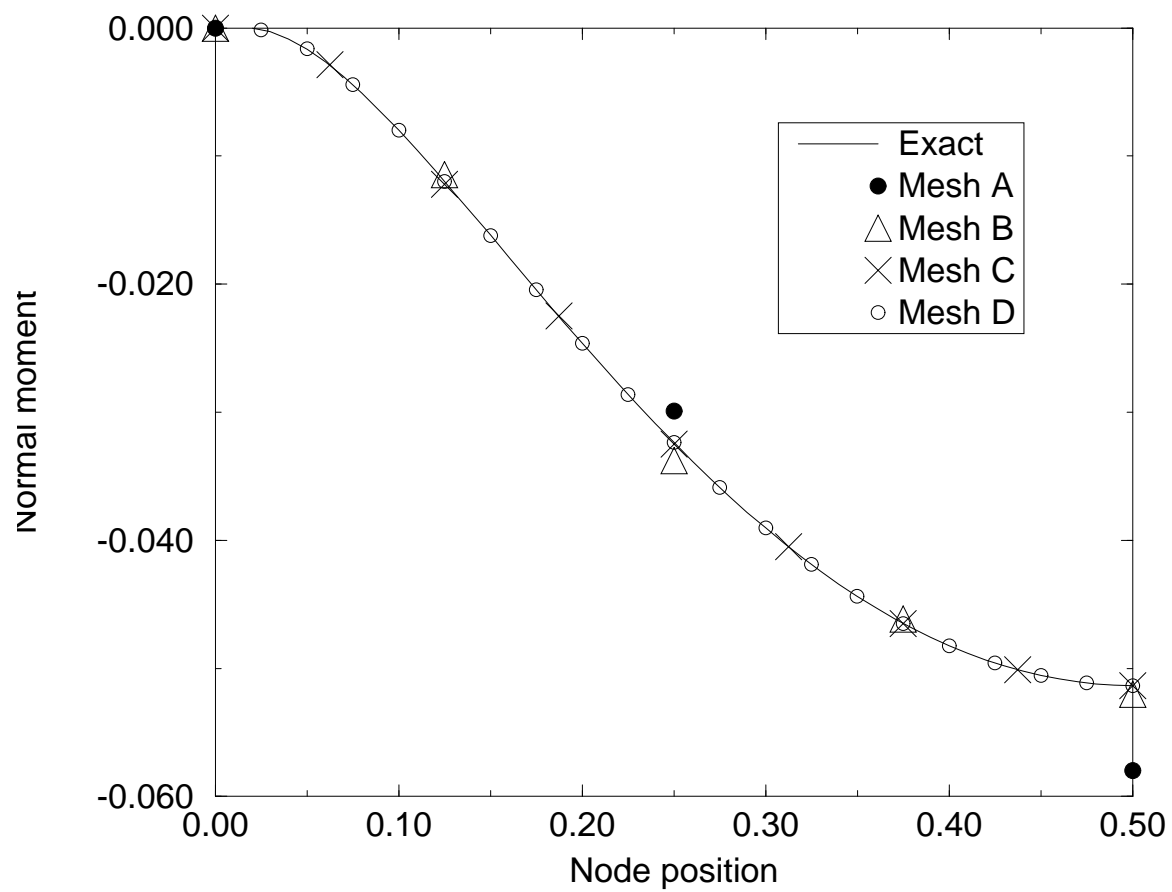


Figure 1: Clamped plate: normal moment.

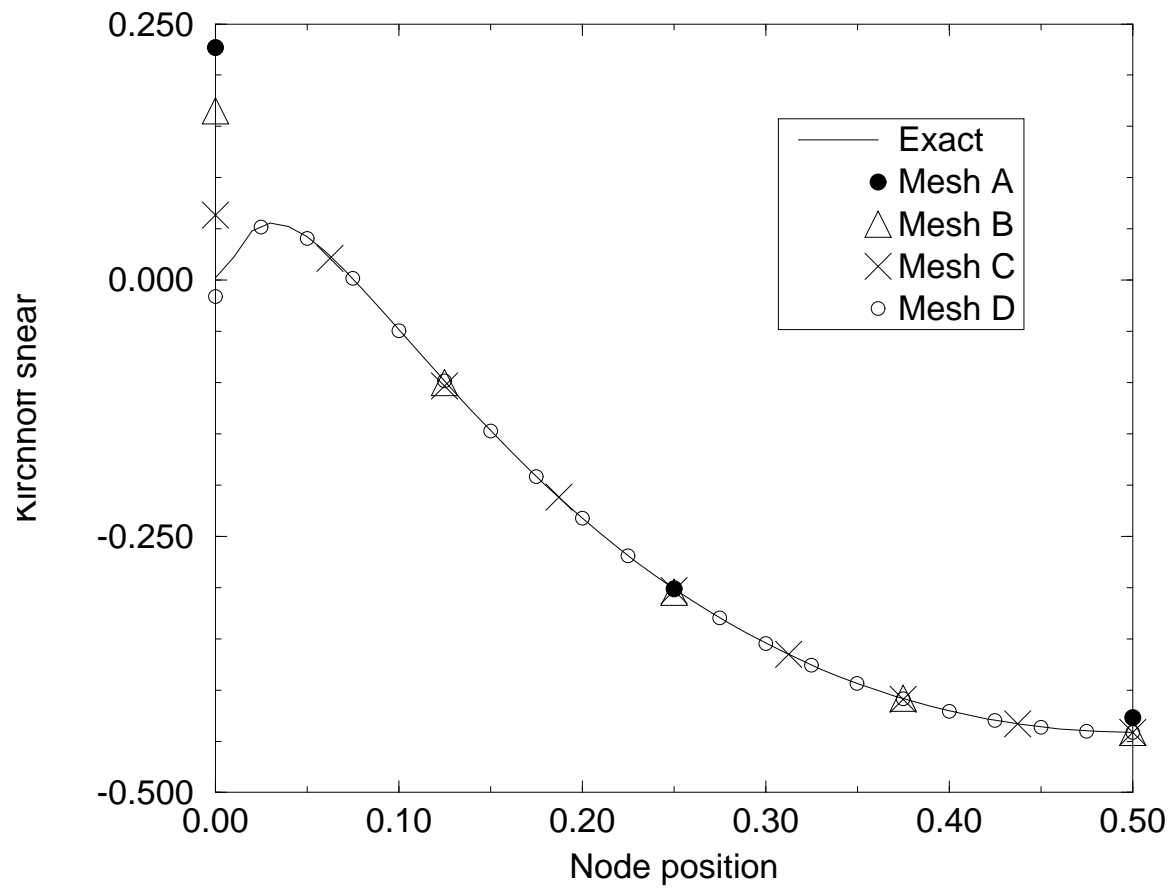


Figure 2: Clamped plate: Kirchhoff shear.

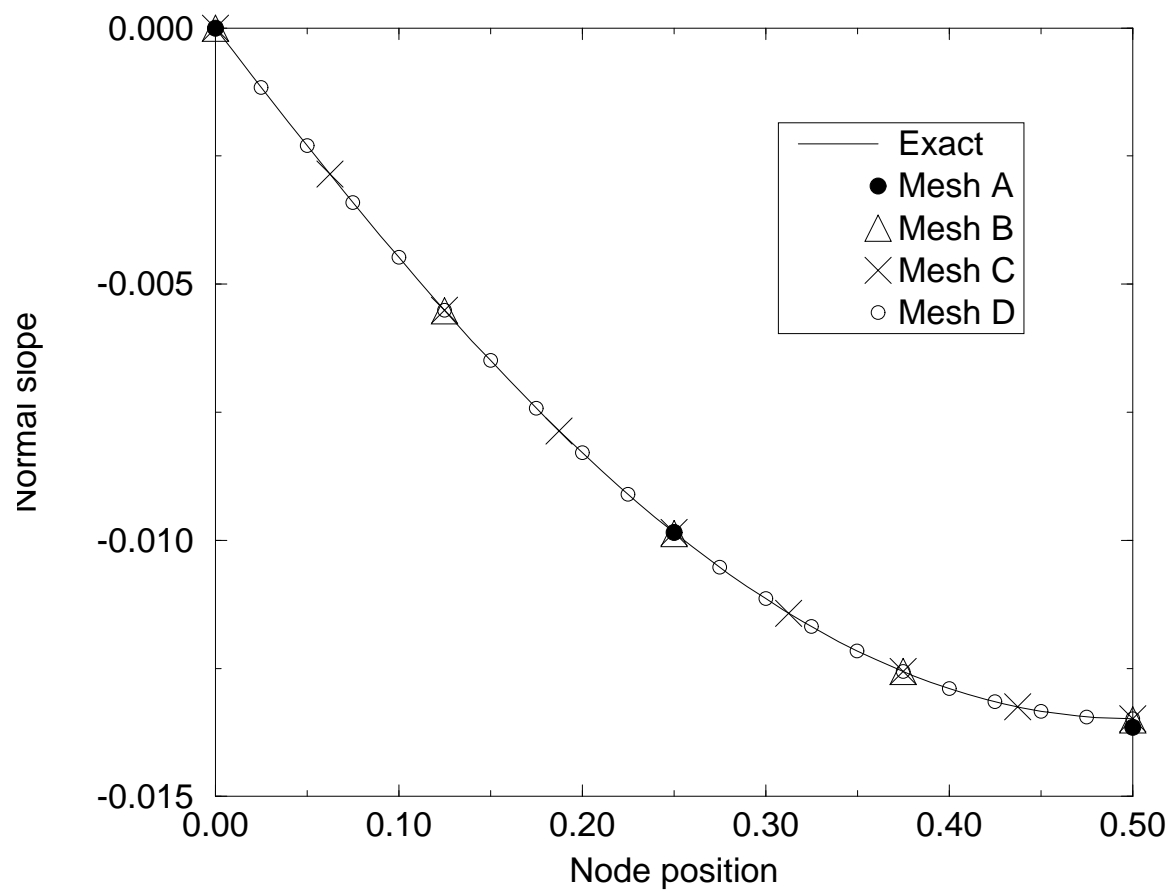


Figure 3: Simply-supported plate: normal slope.

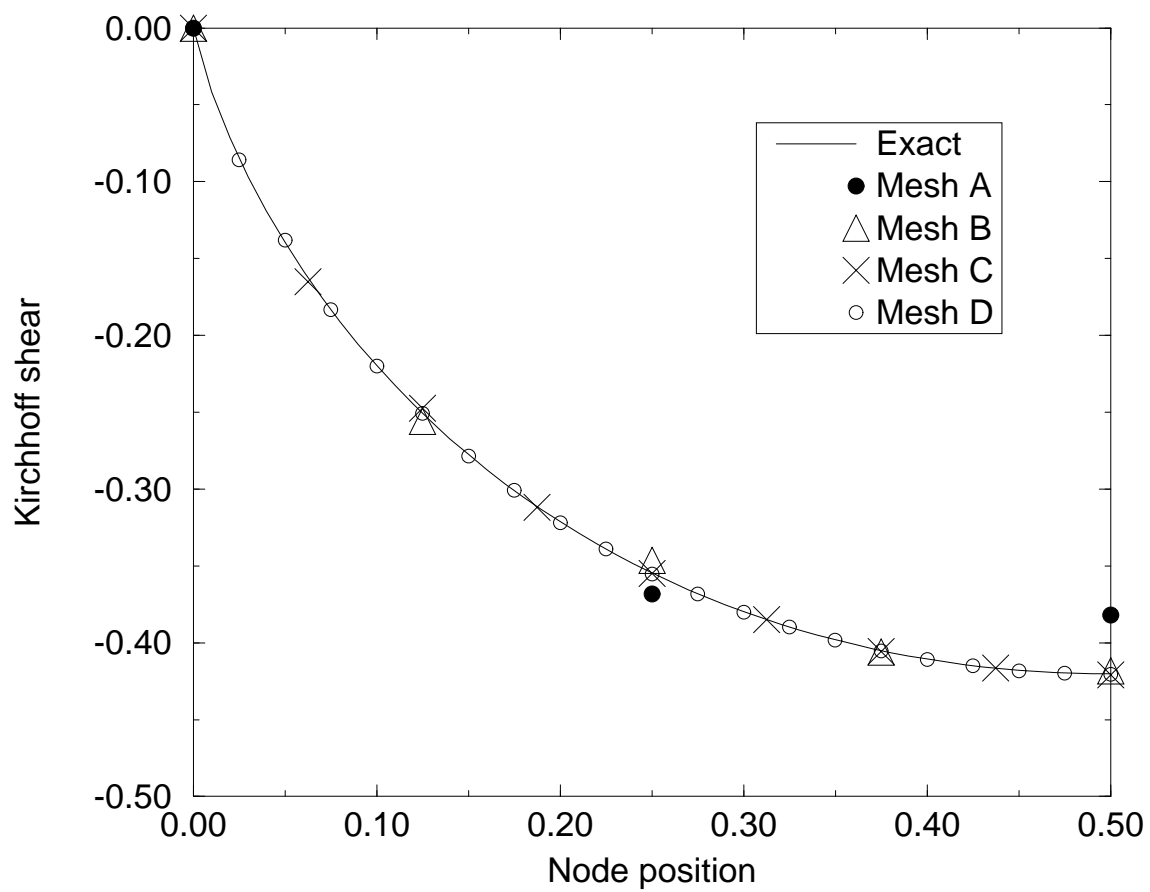


Figure 4: Simply-supported plate: Kirchhoff shear.

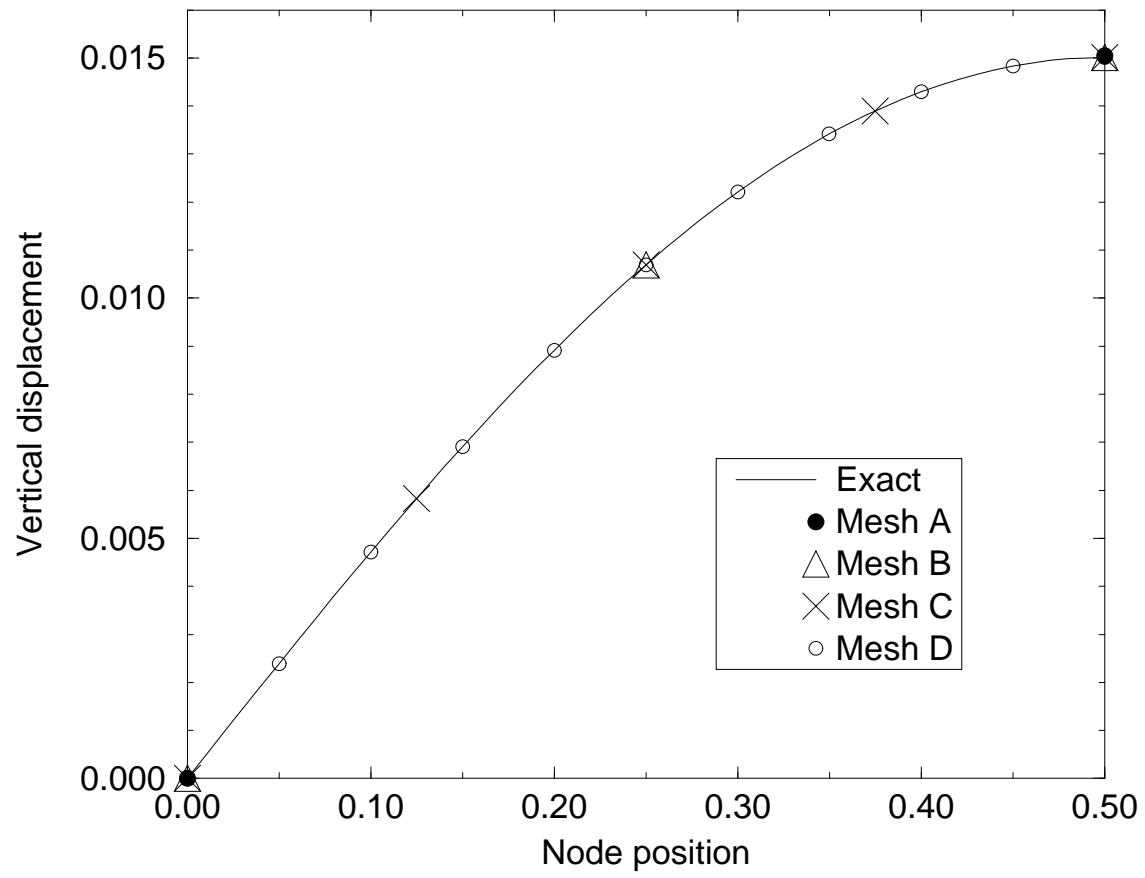


Figure 5: Simply-supported-free plate: vertical displacement along a free side.

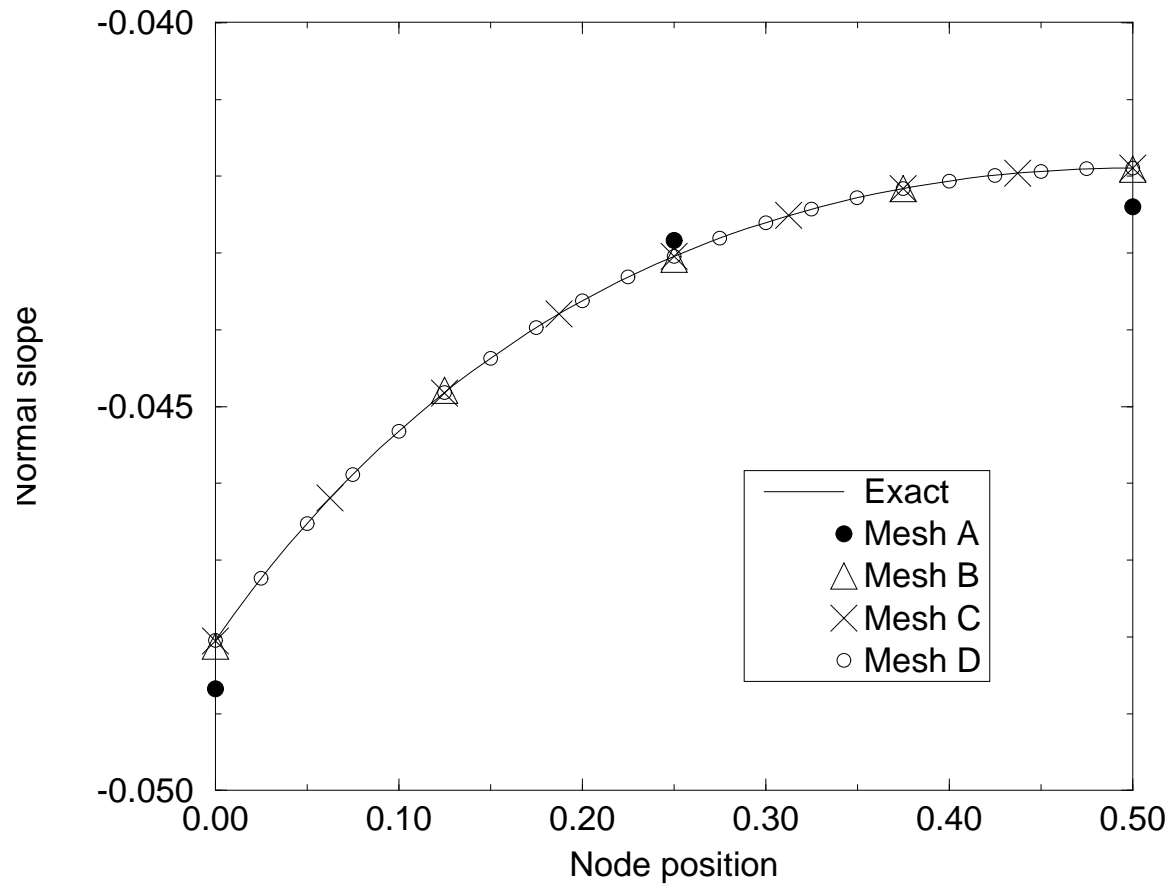


Figure 6: Simply-supported-free plate: normal slope along a simply-supported side.

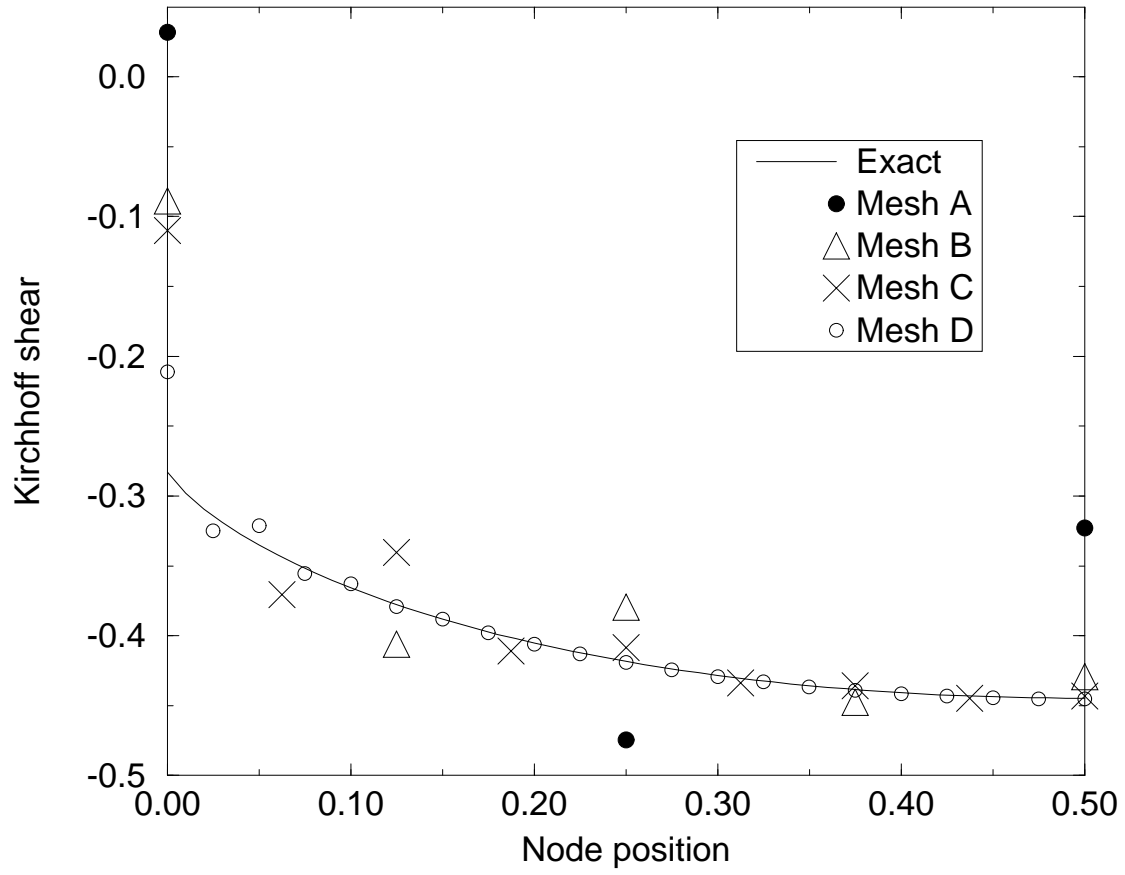


Figure 7: Simply-supported-free plate: Kirchhoff shear along a simply-supported side (3-noded interpolation).

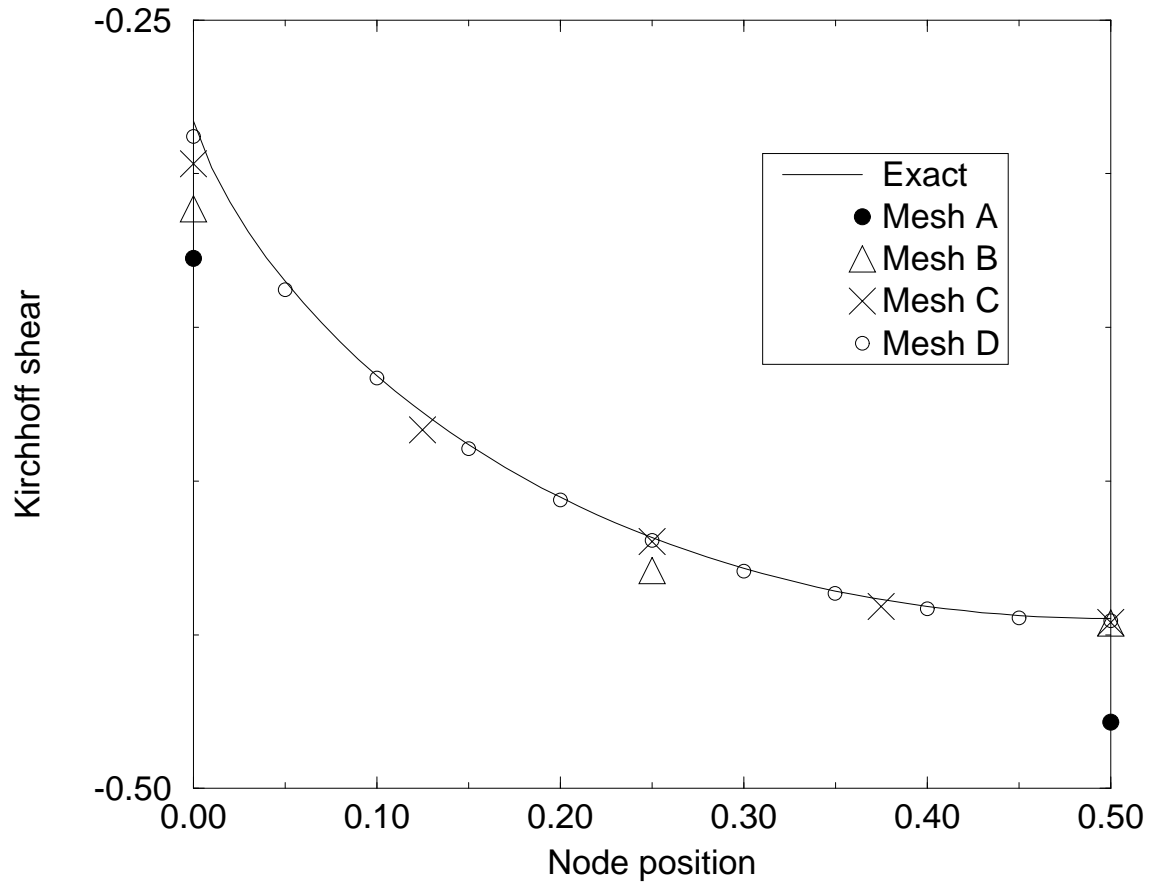


Figure 8: Simply-supported-free plate: Kirchhoff shear along a simply-supported side (2-noded interpolation).

	Mesh	Ref. ¹⁴	present
M_N	A	$.536 \cdot 10^{-1}$.116
	B	$.947 \cdot 10^{-2}$	$.221 \cdot 10^{-1}$
	C	$.132 \cdot 10^{-2}$	$.262 \cdot 10^{-2}$
	D		$.279 \cdot 10^{-3}$
Q_K	A	.634	.421
	B	.306	.240
	C	$.671 \cdot 10^{-1}$	$.675 \cdot 10^{-1}$
	D		$.156 \cdot 10^{-1}$

Table 1: Clamped plate: L^2 relative errors for the normal moment and Kirchhoff shear nodal values along a supported half-side.

Mesh	Ref. ¹⁴	Present
A	$-.31204 \cdot 10^{-1}$	$-.15681 \cdot 10^{-1}$
B	$-.10118 \cdot 10^{-1}$	$-.68198 \cdot 10^{-2}$
C	$-.15677 \cdot 10^{-1}$	$-.15412 \cdot 10^{-2}$
D		$.68153 \cdot 10^{-4}$
	T.&W.-K. ⁴⁴ : 0.	

Table 2: Clamped plate: twisting moment jump

	Mesh	Ref. ¹⁴	present
φ_N	A	$.304 \cdot 10^{-2}$	$.976 \cdot 10^{-2}$
	B	$.819 \cdot 10^{-3}$	$.155 \cdot 10^{-2}$
	C	$.105 \cdot 10^{-3}$	$.286 \cdot 10^{-3}$
	D		$.286 \cdot 10^{-3}$
Q_K	A	$.545 \cdot 10^{-1}$	$.737 \cdot 10^{-1}$
	B	$.104 \cdot 10^{-1}$	$.137 \cdot 10^{-1}$
	C	$.515 \cdot 10^{-2}$	$.472 \cdot 10^{-2}$
	D	$.177 \cdot 10^{-2}$	$.177 \cdot 10^{-2}$

Table 3: Simply-supported plate: L^2 relative errors for the normal slope and Kirchhoff shear nodal values along a supported half-side.

Mesh	Ref. ¹⁴	Present
A	$.58788 \cdot 10^{-1}$	$.59215 \cdot 10^{-1}$
B	$.63592 \cdot 10^{-1}$	$.63330 \cdot 10^{-1}$
C	$.64728 \cdot 10^{-1}$	$.64572 \cdot 10^{-1}$
D		$.64907 \cdot 10^{-1}$
	T.&W.-K. ⁴⁴ :	$.64965 \cdot 10^{-1}$

Table 4: Simply-supported plate: twisting moment jump

	Mesh	Ref. ¹⁴	3-noded Q_K	2-noded Q_K
w	A	$.597 \cdot 10^{-1}$	$.205 \cdot 10^{-2}$	$.226 \cdot 10^{-2}$
	B	$.725 \cdot 10^{-2}$	$.120 \cdot 10^{-3}$	$.160 \cdot 10^{-3}$
	C	$.898 \cdot 10^{-3}$	$.101 \cdot 10^{-3}$	$.833 \cdot 10^{-4}$
	D		$.211 \cdot 10^{-6}$	$.402 \cdot 10^{-6}$
φ_N	A	$.142 \cdot 10^{-1}$	$.790 \cdot 10^{-2}$	$.109 \cdot 10^{-1}$
	B	$.188 \cdot 10^{-2}$	$.571 \cdot 10^{-3}$	$.859 \cdot 10^{-3}$
	C	$.226 \cdot 10^{-3}$	$.332 \cdot 10^{-4}$	$.711 \cdot 10^{-4}$
	D		$.276 \cdot 10^{-4}$	$.278 \cdot 10^{-4}$
Q_K	A	.111	.508	.106
	B	.160	.226	$.456 \cdot 10^{-1}$
	C	$.837 \cdot 10^{-1}$.149	$.173 \cdot 10^{-1}$
	D		$.400 \cdot 10^{-1}$	$.486 \cdot 10^{-2}$

Table 5: Simply-supported-free plate: L^2 relative errors for the bending displacement, normal slope and Kirchhoff shear nodal values along a supported half-side.

Mesh	Ref. ¹⁴	3-noded Q_K	2-noded Q_K
A	$-.49758 \cdot 10^{-1}$	$-.67960 \cdot 10^{-1}$	$-.48859 \cdot 10^{-1}$
B	$-.52573 \cdot 10^{-1}$	$-.54539 \cdot 10^{-1}$	$-.48026 \cdot 10^{-1}$
C	$-.49695 \cdot 10^{-1}$	$-.50800 \cdot 10^{-1}$	$-.48101 \cdot 10^{-1}$
D		$-.48552 \cdot 10^{-1}$	$-.48101 \cdot 10^{-1}$
	T.&W.-K. ⁴⁴ : $-.48089 \cdot 10^{-1}$		

Table 6: Simply-supported-free plate: twisting moment jump

	Equivalent shear force	Normal slope
Exact	-.5	$-.961538 \cdot 10^{-1}$
Numerical	-.499748	$-.961520 \cdot 10^{-1}$

Table 7: Simply-supported circular plate